ON GR*- CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

K. INDIRANI[#1],P. SATHISHMOHAN[#2] AND V. RAJENDRAN[#3]

[#1]*Associate Professor, Department of Mathematics, Nirmala College for Women, Coimbatore, TN, India*

[#2]*Assistant Professor&Head, Department of Mathematics(CA), KSG College of Arts and Science, Coimbatore, TN, India*

[# 3]*Assistant Professor, Department of Mathematics(CA), KSG College of Arts and Science, Coimbatore, TN, India*

ABSTRACT

The aim of this paper is to introduce the concept of generalized regular star continuous and generalized regular star irresolute and study some of its properties.

Keywords: rg-continuous, gr- continuous, gr*-continuous, gr*-irresolute.

1. INTRODUCTION

The concept of regular continuous functions was first introduced by Arya.S.P. and Gupta.R[1]. Later Palaniappan.N. and Rao.K.C.[10] studied the concept of regular generalized continuous functions. Also, the concept of generaliazed regular continuous functions was introduced by Mahmood.S.I.[7]. Recently, the concept of generalized regular star closed set was introduced by Indirani.K. et. al.,[6]. The purpose of this paper is to introduce a new class of functions, namely, generalized regular star continuous functions and generalized regular star irresolute functions. Also, we study some of the characterization and basic properties of generalized regular star continuous functions.

2. PRELIMINARIES

Throughout this paper, (X,τ) (or X) represent a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space X, cl(A) and int(A) denote the closure of A and the interior of A, respectively. X\A or A^c denotes the complement of A in X. We recall the following definitions and results.

Definition 2.1: A map f: $(X,\tau)\rightarrow(Y,\sigma)$ is said to be

(i) g-continuous[2] if $f^{-1}(V)$ is g-closed in X for every closed subset V of Y.

(ii) sg-continuous[11] if $f^{-1}(V)$ is sg-closed in X for every closed subset V of Y.

(iii) gs-continuous[3] if $f^{-1}(V)$ is gs-closed in X for every closed subset V of Y.

(iv) gp-continuous[9] if $f^{-1}(V)$ is gp-closed in X for every closed subset V of Y.

(v) gpr-continuous[5] if $f^{-1}(V)$ is gpr-closed in X for every closed subset V of Y.

(vi) gsp-continuous[4] if $f^{-1}(V)$ is gsp-closed in X for every closed subset V of Y.

(vii) rg-continuous[10] if $f^{-1}(V)$ is rg-closed in X for every closed subset V of Y.

3. gr*- CONTINUOUS FUNCTIONS

Definition 3.1: For a subset A of a space X, gr*-cl(A) = $\bigcap\{F: A\subseteq F, F \text{ is gr* closed in } X\}$ is called the gr*-closure of A.

Definition 3.2 : Let (X,τ) be a topological space and $\tau_{gr*} = \{V \subseteq X: gr^*\text{-}cl(X\backslash V) = X\backslash V\}$.

Lemma 3.3: For any $x \in X$, $x \in gr^*\text{-}cl(A)$ if and only if $V \cap A \neq \phi$, for every gr*-open set V containing X.

Lemma 3.4: Let A and B be subsets of (X,τ). Then
(i) $gr^*\text{-}cl(\phi) = \phi$ and $gr^*\text{-}cl(X) = X$.
(ii) If $A \subseteq B$, then $gr^*\text{-}cl(A) \subseteq gr^*\text{-}cl(B)$.
(iii) $A \subseteq gr^*\text{-}cl(A)$.
(iv) If A is gr*-closed, then $gr^*\text{-}cl(A) = A$.
(v) gr*-closure of a set A is not always gr*-closed.

Remark 3.5: Suppose τ_{gr*} is a topology. If A is gr*-closed in (X,τ), then A is closed in (X, τ_{gr*})

Definition 3.6: A function f from a topological space X into a topological space Y is called a generalixed regular star continuous (gr*-Continuous) if $f^{-1}(V)$ is gr*- Closed set in X for every closed set V in Y.

Theorem 3.7: A function f: $X \to Y$ from a topological space X into a topological space Y is gr*- Continuous iff $f^{-1}(V)$ is gr*- open set in X for every open set V in Y.

Theorem 3.8: If a map f is continuous, then it is gr*- continuous but not converserly.
Proof: Let f: $X \to Y$ be continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is closed set in X. Since every closed set is gr*-closed, $f^{-1}(F)$ is gr*- closed in X. Therefore f is gr*-continuous.

Theorem 3.9: If a map f: $X \to Y$ is r-continuous, then it is gr*- continuous but not converserly.
Proof: Let f: $X \to Y$ be r-continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is r-closed set in X. Since every r-closed set is gr*-closed, $f^{-1}(F)$ is gr*- closed in X. Therefore f is gr*-continuous.

The converse need not be true as seen from the following example.

Example 3.10: Let $X=Y=\{a,b,c,d\}$, $\tau=\{\{\phi\},\{d\},\{bc\},\{bcd\},X\}$ and $\sigma=\{\{\phi\},\{ab\},X\}$. Let f: $X \to Y$ is an identity map, then f is gr*- continuous but not continuous and r-continuous.

Theorem 3.11: If a map f: $X \to Y$ is continuous, Then the following holds.
(i) If f is gr*- continuous, then f is g-continuous.
(ii) If f is gr*continuous, then f is rg-continuous.
(iii) If f is gr*- continuous, then f is gs-continuous.
(iv) If f is gr*- continuous, then f is sg-continuous.
(v) If f is gr*- continuous, then f is gp-continuous.
(vi) If f is gr*- continuous, then f is gsp-continuous.
(vii) If f is gr*- continuous, then f is gpr-continuous.
Proof: (i) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is g-closed, then $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous.
(ii) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is rg-closed, then $f^{-1}(F)$ is rg-closed in X. Hence f is rg-continuous.
(iii) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is gs-closed, then $f^{-1}(F)$ is gs-closed in X. Hence f is gs-continuous.
(iv) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is sg-closed, then $f^{-1}(F)$ is sg-closed in X. Hence f is sg-continuous.
(v) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is gp-closed, then $f^{-1}(F)$ is gp-closed in X. Hence f is gp-continuous.
(vi) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is gsp-closed, then $f^{-1}(F)$ is gsp-closed in X. Hence f is gsp-continuous.

(vii) Let F be a closed set in Y.Since F is gr*-continuous, then $f^{-1}(F)$ is gr*- closed in X. Since every gr*- closed set is gpr-closed, then $f^{-1}(F)$ is gpr-closed in X. Hence f is gpr-continuous.

The converse need not be true as seen from the following example.

Example 3.12: Let X=Y={a,b,c,d},τ={{ ϕ },{ad},{bc},X}and σ={{ ϕ },{a},{b},{ab},X}. Let f: X→Y is an identity map, then f is g- continuous but not gr* –continuous.

Example 3.13: Let X=Y={a,b,c,d},τ={{ ϕ },{d},{bc},{bcd},X}and σ={{ ϕ },{b}X}. Let f: X→Y is an identity map, then f is rg-continuous, gs-continuous, sg-continuous, gp-continuous, gsp-continuous, gpr-continuous but not gr* –continuous.

Theorem 3.14: Let A be a subset of a topological space X. Then x ∈ gr*cl(A) if and only if for any gr*-open set U containing x, A∩U≠ ϕ.

Proof: Let x ∈ gr*cl(A) and suppose that, there is a gr*-open set U in X such that x ∈ U and A∩U≠ ϕ implies that A ⊂ U^c which is gr*-closed in X implies gr*cl(A) ⊆ gr*cl(U^c) = U^c. since x ∈ U implies that x ∉ U^c implies that x ∉ gr*cl(A), this is a contradiction.

Converserly, Suppose that, for any gr*-open set U containing x, A∩U≠ ϕ. To prove that x ∈ gr*cl(A). Suppose that x ∉ gr*cl(A),then there is a gr*-closed set F in X such that x ∉ F and A ⊆ F. Since x ∉ F implies that x ∈ F^c which is gr*-open in X. Since A ⊆ F implies that A ∩ F^c = ϕ, this is a contradiction. Thus x ∈ gr*cl(A).

Theorem 3.15: Let f: X→Y be a function from a topological space X into a topological space Y. If f: X→Y is gr*-continuous, then f(gr*cl(A)) ⊆ cl(f(A)) for every subset A of X.

Proof: Since f(A) ⊆ cl(f(A)) implies that A ⊆ f^{-1}(cl(f(A))). Since cl(f(A)) is a closed set in Y and f is gr*-continuous, then by definition f^{-1}(cl(f(A))) is a gr*-closed set in X containing A. Hence gr*cl(A) ⊆ f^{-1}(cl(f(A))). Therefore f(gr*cl(A)) ⊆ cl(f(A)).

Theorem 3.16: Let f: X→Y be a function from a topological space X into a topological space Y. Then the following statements are equivalent:

(i) For each point x in X and each open set V in Y with f(x) ∈ V, there is a gr*-open set U in X such that x ∈ U and f(U) ⊆ V.

(ii) For each subset A of X, f(gr*cl(A)) ⊆ cl(f(A)).

(iii) For each subset B of Y, gr*cl(f^{-1}(B)) ⊆ f^{-1}(cl(B)).

Proof: (i) → (ii) Suppose that (i) holds and let y ∈ f(gr*cl(A)) and let V be any open neighbourhood of y. Since y ∈ f(gr*cl(A)) implies that there exists x ∈ gr*cl(A) such that f(x) = y. Since f(x) ∈ V, then by (i) there exists a gr*-open set U in X such that x ∈ U and f(U) ⊆ V. Since x ∈ f(gr*cl(A)), then by theorem 3.14 U∩A≠ϕ. Therefore we have y = f(x) ∈ cl(f((A)). Hence f(gr*cl(A)) ⊆ cl(f(A)).

(ii) → (i) Let if (ii) holds and let x ∈ X and V be any pen set in Y containing f(x). Let A = $f^{-1}(V^c)$ this implies that x ∉ A. Since f(gr*cl(A)) ⊆ cl(f(A)) ⊆ V^c this implies that gr*cl(A) ⊆ $f^{-1}(V^c)$ = A. Since x ∉ A implies that x ∉ gr*cl(A) and by theorem 3.14 there exists a gr*-open set U containing x such that U∩A≠ ϕ and hence f(U) ⊆ f(A^c) ⊆ V.

(ii) → (iii) Suppose that (ii) holds and Let B be any subset of Y. Replacing A by f^{-1}(B) we get from (ii) f(gr*cl(f^{-1}(B))) ⊆ cl(f(f^{-1}(B))) ⊆ cl(B). Hence gr*cl(f^{-1}(B)) ⊆ f^{-1}(cl(B)).

(iii) → (ii) Suppose that (iii) holds, let B = f(A) where A is a subset of X. Then we get from (iii) gr*cl(A) ⊆ gr*cl(f^{-1}(f(A)) ⊆ f^{-1}(cl(f(A))). Therefore f(gr*cl(A)) ⊆ cl(f(A)).

Theorem 3.17: Let f: X→Y be a map. Then the following statements are equivalent :

(i) f is gr*-continuous.

(ii) the inverse image of each open set in y is gr*-open in x.

Proof: Assume that f: $X \rightarrow Y$ is gr*-continuous. Let G be open in y. The G^c is closed in y. Since f is gr*-continuous, $f^{-1}(G^c)$ is gr*-closed in X. But $f^{-1}(G^c) = X - f^{-1}(G)$. Thus $f^{-1}(G)$ is gr*-open in X.

Converserly assume that the inverse image of each open set in Y is gr*-open in X. Let F be any closed set in Y. By assumption c is gr*-open in X. But $f^{-1}(F^c) = X - f^{-1}(F)$. Thus $X - f^{-1}(F)$ is gr*-open in X and so $f^{-1}(F)$ is gr*-closed in X. Therefore f is gr*-continuous. Hence (i) and (ii) are equivalent.

Theorem 3.18: If a function f: $X \rightarrow Y$ is gr*-continuous, then f(gr*-cl(A)) ⊆ cl(f(A)) for every subset A of X.

Proof: Let f: $X \rightarrow Y$ be gr*-continuous. Let A⊆X. Then cl(f(A)) is closed inY. Since f is gr*-continuous, f^{-1}(cl(f(A))) is gr*-closed in X and A⊆ f^{-1}(f(A)) ⊆ f^{-1}(cl(f(A))),implies gr*-cl(A) ⊆ f^{-1}(cl(f(A))). Hence f(gr*-cl(A)) ⊆ cl(f(A)).

Theorem 3.19: Let f: $X \rightarrow Y$ be a function. Let (X,τ) and (Y,σ) be any two spaces such that τ_{gr*} is a topology on X. Then the following statements are equivalent:

(i) For every subset A of X, f(gr*-cl(A)) ⊆ cl(f(A)) holds,

(ii) f: $(X, \tau_{gr*}) \rightarrow (Y,\sigma)$ is continuous.

Proof: Suppose (i) holds. Let A be closed in Y. By hypothesis f(gr*-cl(f^{-1}(A))) ⊆ cl(f(f 1(A))) ⊆ cl(f(A)) = A. i.e., gr*-cl(f^{-1}(A)) ⊆ f^{-1}(A). Also f^{-1}(A) ⊆ gr*-cl(f^{-1}(A)). Hence, gr*- cl(f^{-1}(A)) = f^{-1}(A). This implies $(f^{-1}(A))^c \in \tau_{gr*}$. Thus f^{-1}(A) is closed in (X, τ_{gr*}) and so f is continuous. This proves (ii).

Suppose (ii) holds. For every subset A of X, cl(f(A)) is closed in Y. Since f: $(X, \tau_{gr*}) \rightarrow (Y,\sigma)$ is continuous, f^{-1}(cl(A))) is closed in (X, τ_{gr*}) that implies by Definition 3.2 gr*-cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))). Now we have, A ⊆ f^{-1}(f(A)) ⊆ f^{-1}(cl(f(A))) and by lemma 3.4 (ii) , gr-cl(A) ⊆ gr*-cl(f^{-1}(cl(f(A))) = f^{-1}(cl(f(A)). Therefore f(gr*-cl(A)) ⊆ cl(f(A)).

Theorem 3.20: Let f: $X \rightarrow Y$ is gr*-continuous function and g:$Y \rightarrow Z$ is continuous function then g∘f: $X \rightarrow Z$ is gr*-continuous.

Proof: Let g be a continuous function and V be any open set in Z then f^{-1}(V) is open in Y. Since f is gr*-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is gr*-open in X. Hence g∘f is gr*-continuous.

Theorem 3.21: Let X = A∪B be a topological space with topology τ and Y be a topological space with topology σ. Let f: $(A, \tau/A) \rightarrow (Y,\sigma)$ and g : $(B, \tau/B) \rightarrow (Y,\sigma)$ be gr*-continuous maps such that f(x) = g(x) for every x∈A∩B. Suppose that A and B are gr*-closed sets in X. Then the combination $\alpha : (X,\tau) \rightarrow (Y,\sigma)$ is gr*-continuous.

Proof: Let F be any closed set in Y. Clearly $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ = C∪D where C = f^{-1}(F) and D = g^{-1}(F). But C is gr*-closed in A and A is gr*-closed in X and so C is gr*-closed in X. Since we have proved that if B ⊆ A ⊆ X, B is gr*-closed in A and A is gr*-closed in X then B is gr*-closed in X. Also C∪D is gr*-closed in X. Therefore α^{-1}(F) is gr*-closed in X. Hence α is gr*-continuous.

Definition 3.22: A function f: $X \rightarrow Y$ is said to be gr*-irresolute, if f^{-1}(V) is gr*-open set in (X,τ) for every g*r-open set V in (Y,σ).

Theorem 3.23: Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be any two functions. Let h = g∘f. Then:

(i) h is gr*-continuous if f is gr*-irresolute and g is gr*-continuous and

(ii) h is gr*-continuous if g is continuous and f is gr*-continuous.

Proof: Let V be closed in Z. (i) Suppose f is gr*-irresolute and g is gr*-continuous. Since g continuous is gr*-continuous. g^{-1}(V) is gr*-closed in Y. Since f is gr*-irresolute, using the Definition 3.22 $f^{-1}(g^{-1}(V))$ is gr*-closed in X. This proves (i).

(ii) Let g be continuous and f be gr*-continuous. Then g^{-1}(V) is closed in Y. Since f is gr*-

continuous, using the Definition 3.6 $f^{1}(g^{-1}(V))$ is gr*-closed in X. This proves (iii).

Theorem 3.24: A function f: X→Y is gr*-irresolute if and only if the inverse image of every gr*-open set in Y is gr*-open in X.

Definition 3.25: A function f: X→Y is said to be gr*-closed (resp. gr*-open) if for every gr*-closed (resp. gr*-open) set U of X the set f(U) is gr*-closed (resp. gr*-open) in Y.

Theorem 3.26: A function f: X→Y be a bijection. Then the following are equivalent:

(i) f is gr*-open,

(ii) f is gr*-closed,

(iii) f^{1} is gr*-irresolute.

Proof: Suppose f is gr*-open. Let F be gr*-closed in X. Then X\F is gr*-open. By Definition 3.25 f(X\F) is gr*-open. Since f is bijection, Y\f(F) is gr*-open in Y. Therefore f is gr*-closed. This proves (i) ⇒(ii).

Let g = f^{1}. Suppose f is gr*-closed. Let V be gr*-open in X. Then X\V is gr*-closed in X. Since f is gr*-closed, f(X\V) is gr*-closed. Since f is a bijection, Y\f(V) is gr*-closed that implies f(V) is gr*-open in Y. Since g = $f^{1}$and since g and f are bijection $g^{-1}(V)$ = f(V) so that $g^{-1}(V)$ is gr*-open in Y. Therefore f^{1} is gr*-irresolute. This proves (ii) ⇒(iii).

Suppose f^{1} is gr*-irresolute. Let V be gr*-open in X. Then X\V is gr*-closed in X. Since f^{1} is gr*-irresolute and $(f^{1})^{-1}(X\backslash V) = f(X\backslash V) = Y\backslash f(V)$ is gr*-closed in Y that implies f(V) is gr*-open in Y. Therefore f is gr*-open. This proves (iii) ⇒(i).

Theorem 3.27: Let f: X→Y and g: Y→Z are gr*-irresolute, then g∘f: X→Z is gr*-irresolute.

Proof: Let g be an gr*-irresolute function and V be any gr*-open in Z, then $f^{1}(V)$ is gr*-open set in Y, since f is gr*-irresolute, $f^{1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is gr*-open in (X,τ). Hence g∘f is gr*-irresolute.

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