COMPATIBLE MAPPINGS OF TYPE - (I) AND (II) ON QUASI- GAUGE SPACE IN CONSIDERATION OF COMMON FIXED POINT

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ABSTRACT

In this paper, we first formulate the definitions of compatible mappings of Type- (I) and (II) in the settings of Quasi- Gauge Space the purpose of this paper is to prove common fixed point theorems for four mappings by using the condition of compatible mappings of Type- (I) and (II) in Quasi-Gauge Space. Which generalizes earlier results of Rao and Murti [4]. Our result has a number of applications in mathematical sciences.

KEYWORDS: Common fixed point, compatible maps, compatible mappings of type (I) and (II), Quasi-gauge space.

I. INTRODUCTION

Pathak, Chang and Cho [3] proved fixed point theorems for compatible mappings of type (P). Rao and Murthy [4] extended results on common fixed points of self maps by replacing the domain"complete metric space" with "Quasi-gauge space". But in both theorems continuity of any mapping was the necessary condition for the existence of the fixed point. I generalize earlier results of Rao and Murthy [4] and show that common fixed point theorems for four mappings of compatible mappings of Type- (I) and (II) in Quasi-Gauge Space. by using a S.Sharma, B.Deshpande and R.Pathak, [6] Compatible Mappings of Type- (I) and (II) On Intuitionistic Menger Spaces In Consideration Of Common Fixed Point Our result has a number of applications in mathematical sciences.

Definition 1.1 A Quasi-pseudo-metric on a set X is a non negative real valued function p on XxX such that

(i) $p(x, x) = 0$ for all $x \in X$.

(ii) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$.

Definition 1.2 A Quasi-gauge structure for a topological space (X,T) is a family P of quasi-pseudo-metrics on X such that T has as a sub-base the family

$$\{B(x, p, \varepsilon) : x \in X, p \in P, \varepsilon > 0\},$$

where $B(x, p, \varepsilon)$ is the set $\{y \in X: p(x, y) < \varepsilon\}$. If a topological space has a Quasi-gauge structure, it is called a quasi-gauge space.

Definition 1.3 A sequence $\{x_n\}$ in a Quasi-gauge space (X, P) is said to be P-Cauchy, if for each $\varepsilon>0$ and $p \in P$ there is an integer k such that $p(x_m, x_n) < \varepsilon$ for all $m, n \geq k$.

Definition 1.4 A Quasi-gauge space (X,P) is sequentially complete if and only if every P-Cauchy sequence in X is convergent in X. We now propose the following characterization. Let (X, P) be a Quasi-gauge space X is a T_0 Space if and only if $p(x, y) = 0$, for all p in P implies $x = y$.

Definition 1.5 Let A, S be mappings from a Quasi Gauge space. Then the pair (A, S) is compatible of type (I) if for all $t > 0$. Letting $\lim_{n\to\infty} SSx_n = Sz$, whenever $\{x_n\}$ is a sequence in X such that

$$p(Sz, z) \leq \lim_{n\to\infty} p(ASx_n, z)$$

Definition 1.6 Let A, S be mappings from a Quasi Gauge space. Then the pair (A, S) is compatible of type (II) if and only if (S, A) compatible of type (I).

Lemma 1.1 Suppose that $\psi: [0, \infty) \to [0, \infty)$ is non-decreasing and upper semi-continuous from the right. If $\psi(t) < t$ for every $t > 0$, then $\lim \psi^n(t) = 0$.

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II. PRELIMINARIES

Rao and Murty [4] proved the following.

Theorem 2.1 Let A, B, S and T be self maps on a left(right) sequentially complete Quasi-gauge T_0 space (X,P) such that

(i) (A, S), (B, T) are weakly compatible pairs of maps with $T(X) \subseteq A(X)$; $S(X) \subseteq B(X)$;

(ii) A and B are continuous;

(iii) $\max\{p^2(Sx, Ty), p^2(Ty, Sx)\} \leq ø\{p(Ax, Sx)p(By, Ty), p(Ax, Ty)p(By ,Sx), p(Ax, SSx)p(Ax, Ty), p(By, Sx)p(By, Ty), p(By, Sx)p(Ax, Sx), p(By, Ty) p(Ax, Ty)\}$;

for all x, y ∈ X and for all p in P, where $ø : [0, \infty)^6 \to (0, +\infty)$ satisfies the following

(iv) ø is non-decreasing and upper semi-continuous in each coordinate variable and for each t > 0

$$\Psi(t)=\max\{ø(t,0,2t,0,0,2t),ø(t,0,0,2t,2t,0),ø(0,t,0,0,0,0),ø(0,0,0,0,0,t)\ ø(0,0,0,0,t,0)\} < t;$$

Then A, B, S and T have a unique common fixed point.

Theorem 2.2 Let A, B, S and T be self maps on a left(right) sequentially complete Quasi-gauge T_0 space (X, P) with condition (iii) and (iv) of Theorem 2.1 such that

(i) (S, A), (A, S), (B, T) and (T, B) are weakly compatible pairs of maps with $T(X) \subseteq A(X)$; $S(X) \subseteq B(X)$;

(ii) One of A, B, S and T is continuous:

Then the same conclusion of Theorem 2.1 holds.

We prove Theorem 2.1 and Theorem 2.2 without assuming that any function is continuous and proof a theorem of finite number of mapping (increasing mappings).

III. Main results

We prove the following:

Theorem 3.1 Let A, B, S and T be self maps on a left(right) sequentially complete Quasi-gauge T_0 Space (X, P) such that

(3.1.1) the pairs (A, S) and (B, T) are compatible of type (II) and A or B is continuous,

(3.1.2) the pairs (A, S) and (B, T) are compatible of type (I) and S or T is continuous,

$T(X) \subseteq A(X)$, $S(X) \subseteq B(X)$;

$\max\{p^2(Sx, Ty),p^2(Ty, Sx)\} \leq ø\{p(Ax, Sx)p(By, Ty), p(Ax ,Ty)p(By, Sx),p(Ax, Sx)p(Ax, Ty),p(By, Sx)p(By, Ty), p(By, Sx)p(Ax ,Sx), p(By, Ty)p(Ax, Ty)\}$;

for all x, y ∈X and for all p in P, where $ø : [0, \infty)^6 \to (0, +\infty)$ satisfies the following:

(3.1.3) ø is non-decreasing and upper semi-continuous in each coordinate variable and for each t >0:

$\Psi(t) = \max\{ø(t,0,2t,0,0,2t), ø(t,0,0,2t,2t,0), ø(0,t,0,0,0,0), ø(0,0,0,0,0,t)\ ø(0,0,0,0,t,0)\} < t$.

then the mappings A, B, S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. since (3.1.1) holds we can choose x_1, x_2 in X such that $Bx_1 = Sx_0$ and $Ax_2 = Tx_1$ in general we can choose x_{2n+1} and x_{2n+2} in X such that

(3.1.4) $y_{2n} = Bx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Ax_{2n+2}= Tx_{2n+1}$; n = 0,1,2.....

We denote $d_n= p(y_n, y_{n+1})$ and $e_n = p(y_{n+1}, y_n)$; now applying (3.1.2) we get

$\max\{d^2_{2n+2}, e^2_{2n+2}\} = \max\{p^2(Sx_{2n+2}, Tx_{2n+3}), p^2(Tx_{2n+3}, Sx_{2n+2})\}$

$\leq ø\{p(Ax_{2n+2}, Sx_{2n+2})p(Bx_{2n+3},Tx_{2n+3}), p(Ax_{2n+2}, Tx_{2n+3})p(Bx_{2n+3}, Sx_{2n+2}), p(Ax_{2n+1}, Sx_{2n+2})p(Ax_{2n+2},Tx_{2n+1}), p(Bx_{2n+1},Sx_{2n+2})p(Bx_{2n+2},Tx_{2n+3},), p(Bx_{2n+3}, Sx_{2n+2})\ p(Ax_{2n+2}, Sx_{2n+1}), p(Bx_{2n+3}, Tx_{2n+3})\ p(Ax_{2n+2}, Tx_{2n+3})\}$;

$=ø\{p(y_{2n+1} ,y_{2n+2})p(y_{2n+2}, y_{2n+3}), p(y_{2n+1}, y_{2n+3})p(y_{2n+2}, y_{2n+2}), p(y_{2n+1}, y_{2n+2},)p(y_{2n+1}, y_{2n+3}), p(y_{2n+2}, y_{2n+2})p(y_{2n+2}, y_{2n+3},), p(y_{2n+2}, y_{2n+2})p(y_{2n+1}, y_{2n+2}), p(y_{2n+2}, y_{2n+3})p(y_{2n+1}, y_{2n+3})\}$

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$\leq ø\{ d_{2n+1}, d_{2n+2}, 0, d_{2n+1}(d_{2n+1} + d_{2n+2}), 0, 0, d_{2n+2} (d_{2n+1} + d_{2n+2})\}$

(3.1.5) If $d_{2n+2} > d_{2n+1}$ then

(3.1.6) $\max \{d^2_{2n+2}, e^2_{2n+2}\} \leq ø\{ d^2_{2n+2}, 0, 2d^2_{2n+2}, 0, 0, 2d^2_{2n+2}\} < d^2_{2n+2}$.

by (3.1.3) a contradiction; hence $d_{2n+2} \leq d_{2n+1}$.

Similarly we get

(3.1.7) $d_{2n+1} \leq d_{2n}$.

By (3.1.5) and (3.1.6)

$\max\{d^2_{2n+2}, e^2_{2n+2}\} \leq ø\{ d^2_{2n+1}, 0, 2d^2_{2n+1}, 0, 0, 2d^2_{2n+1}\} \leq \Psi(d^2_{2n+1})$

(3.1.8) $= \Psi\{P^2(y_{2n+1}, y_{2n+2})\}$.

Similarly we have,

$\max\{d^2_{2n+1}, e^2_{2n+1}\} \leq ø \{ d^2_{2n}, 0, 0, 2d^2_{2n}, 2d^2_{2n}, 0\}$.

(3.1.9) $\leq \Psi \{P^2(y_{2n}, y_{2n+1})\}$.

So

(3.1.10) $d^2_n = P^2(y_n, y_{n+1}) \leq \Psi \{P^2(y_{n+1}, y_n)\} \leq . . . \leq \Psi^{n-1}\{P^2(y_1, y_2)\}$

and

(3.1.11) $e^2_n = P^2(y_{n+1}, y_n) \leq \Psi\{P^2(y_{n-1}, y_n)\} \leq . . . \leq \Psi^{n-1}\{P^2(y_1, y_2)\}$

Hence by Lemma 1.1 and from (3.1.10) and (3.1.11) we obtain

(3.1.12) $\lim d_n = e_n = 0$.

Now we prove $\{y_n\}$ is a P-Cauchy sequence. To show $\{y_n\}$ is P-Cauchy it is enough if we show $\{y_{2n}\}$ is P-Cauchy. Suppose $\{y_{2n}\}$ is not a P-Cauchy sequence then there is an $\varepsilon > 0$ such that for each positive integer 2k there exist positive integers 2m(k) and 2n(k) such that for some p in P,

(3.1.13) $p(y_{2n(k)}, y_{2m(k)}) > \varepsilon$ for $2m(k) > 2n(k) > 2k$

and

(3.1.14) $p(y_{2m(k)}, y_{2n(k)}) > \varepsilon$ for $2m(k) > 2n(k) > 2k$

for each positive even integer 2k, let 2m(k) be the least positive even integer exceeding 2n(k) and satisfying (3.1.13); hence $p(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon$ then for each even integer 2k,

$\varepsilon < p(y_{2n}(k), y_{2m}(k)$

(3.1.15) $\leq p(y_{2n(k)}, y_{2m(k)-2}) + (d_{2m(k)-2} + d_{2m(k)-1})$

From (3.1.12) and (3.1.15), we obtain $\lim p(y_{2n(k)}, y_{2m(k)}) = \varepsilon$. By the triangle inequality

$p(y_{2n(k)}, y_{2m(k)}) \leq p(y_{2n(K)}, y_{2m(k)-1}) + d_{2m(k)-1}\}p(y_{2n(K)}, y_{2m(k)-1}) \leq p(y_{2n(k)}, y_{2m(k)}) + e_{2m(k)-1}$;

So

(3.1.16) $|p(y_{2n(k)}, y_{2m(k)})-p(y_{2n(k)}, y_{2m(k)-1})| \leq \max\{ d_{2m(k)-1}, e_{2m(k)-1}\}$.

Similarly By triangle inequality

(3.1.17) $|p(y_{2n(K)+1}, y_{2m(k)-1}) - p(y_{2n(K)}, y_{2m(k)})| \leq \max\{e_{2n(k)} + e_{2m(k)-1}, d_{2n(k)} + d_{2m(k)-1}\}$

From (3.1.16) and (3.1.17) as $k\to\infty$, $\{p(y_{2n(K)}, y_{2m(k)-1})\}$ and $p(y_{2n(K)+1}, y_{2m(k)-1})\}$ converge to ε. Similarly if $p(y_{2m(K)}, y_{2n(k)}) > \varepsilon$,

$\lim p(y_{2m(K)}, y_{2n(k)}) = \lim p(y_{2m(K)-1}, y_{2n(k)+1})$

$= \lim p(y_{2m(K)-1}, y_{2n(k)}) = \varepsilon$ as $k\to\infty$.

By (3.1.2)

$\varepsilon < p(y_{2n(K)}, y_{2m(k)})$

$\leq p(y_{2n(K)}, y_{2n(k)+1}) + p(y_{2n(K)+1}, y_{2m(k)})$

$\leq d_{2n(k)} + \max\{ p(y_{2n(K)+1}, y_{2m(k)}), p(y_{2n(K)}, y_{2n(k)+1})\}$

$= d_{2n(k)}+\max\{p(Tx_{2n(k)+1}, Sx_{2m(k)}), p(Sx_{2m(k)}, Tx_{2n(k)+1})\}$

$\leq d_{2n(k)}+[ø\{p(y_{2m(K)-1}, y_{2m(k)})p(y_{2n(K)}, y_{2n(k)+1}), p(y_{2m(K)-1}, y_{2n(k)+1})p(y_{2n(K)}, y_{2m(k)}), p(y_{2m(K)-1}, y_{2m(k)})p(y_{2m(K)-1}, y_{2n(k)+1}), p(y_{2n(K)}, y_{2m(k)})p(y_{2n(K)}, y_{2n(k)+1}), p(y_{2n(K)}, y_{2m(k)})p(y_{2m(K)-1}, y_{2m(k)}), p(y_{2n(K)}, y_{2n(k)+1})p(y_{2m(K)-1}, y_{2n(k)+1})\}]^{1/2}$

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Since ø is upper semi-continuous, as k→∞ we get that $\varepsilon \le \{ø(0, \varepsilon^2, 0, 0, 0, 0)\}^{1/2} < \varepsilon$, which is a contradiction. Therefore $\{y_n\}$ is P-Cauchy sequence in X. Since X is complete there exists a point z in X such that lim n→∞ $y_n=z$.

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n-1} = z$$

and

$$\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n-2} = z$$

Now suppose that S is continuous and the pair (A, S) is compatible of type (I). Hence we have

Letting $\lim_{n\to\infty} SSx_{2n} = Sz$, $p(Sz, z) \le \lim_{n\to\infty} p(ASx_{2n}, z)$

Setting $x = Sx_{2n}$, $y = x_{2n+1}$ in the inequality (3.1.2).

$$\max\{p^2(SSx_{2n},Tx_{2n+1}), p^2(Tx_{2n+1},SSx_{2n})\}$$

$$\le ø\{p(ASx_{2n}, SSx_{2n})p(Bx_{2n+1}, Tx_{2n+1}), p(ASx_{2n}, Tx_{2n+1})p(Bx_{2n+1}, SSx_{2n}), p(ASx_{2n}, SSx_{2n})p(ASx_{2n}, Tx_{2n+1}), p(Bx_{2n+1}, SSx_{2n})p(Bx_{2n+1}, Tx_{2n+1}), p(Bx_{2n+1}, SSx_{2n})p(ASx_{2n}, SSx_{2n}), p(Bx_{2n+1}, Tx_{2n+1})p(ASx_{2n}, Tx_{2n+1})\};$$

Letting lim n→∞,we have

$$\max\{p^2(Sz, z), p^2(z, Sz)\}$$

$$\le ø\{p(Sz, Sz)p(z, z), p(Sz, z)p(z, Sz), p(Sz, Sz)p(Sz, z), p(z, Sz)p(z, z), p(z, Sz)p(Sz, Sz), p(z, z)p(Sz, z)\};$$

$$\le ø(0, p(Sz, z)p(z, Sz), 0, 0, 0, 0)$$

A contradiction Sz = z.

Again replacing x by z and y by Tx_{2n+1} in the inequality (3.1.2) we have

$$\max\{p^2(Sz,TTx_{2n+1}), p^2(TTx_{2n+1},Sz)\}$$

$$\le ø\{p(Az, Sz)p(BTx_{2n+1}, TTx_{2n+1}), p(Az, TTx_{2n+1})p(BTx_{2n+1}, Sz), p(Az, Sz)p(Az, TTx_{2n+1}), p(BTx_{2n+1}, Sz)p(BTx_{2n+1}, TTx_{2n+1}), p(BTx_{2n+1}, Sz)p(Az, Sz), p(BTx_{2n+1}, TTx_{2n+1})p(Az, TTx_{2n+1})\};$$

Letting lim n→∞

$$\max\{p^2(z, Tz), p^2(Tz, z)\}$$

$$\le ø \{p(z, z)p(Tz, Tz), p(z, Tz)p(Tz, z), p(z, z)p(z, Tz), p(Tz, z)p(Tz, Tz), p(Tz, z)p(z, z), p(Tz, Tz)p(z, Tz)\};$$

$$\le ø (0, p(z, Tz), 0, 0, 0, 0)$$

A contradiction Tz = z.

Since S(X) ⊆ B(X), there exist a point u∈ X such that Bu = z = Sz. So by (3.1.1), we have,

$$\max\{p^2(Sz,Tu), p^2(Tu,Sz)\}$$

$$\le ø\{p(Az, Sz)p(Bu, Tu), p(Az, Tu)p(Bu, Sz), p(Az,Sz)p(Az,Tu), p(Bu, Sz)p(Bu, Tu), p(Bu, Sz)p(Az ,Sz), p(Bu, Tu)p(Az, Tu)\}.$$

Taking as limn→∞,

$$\max\{p^2(z, Tu), p^2(Tu, z)\}$$

$$\le ø\{p(z, z)p(z, Tu), p(z, Tu)p(z, z), p(z, z)p(z, Tu), p(z, z)p(z, Tu), p(z, z)p(z, z), p(z, Tu)p(z, Tu)\}$$

Hence Tu = z since the pair (B, T) is compatible of type (I) and Tu = Bu = z by Therefore z is a common fixed point of the self mappings A, B, S and T.

The uniqueness of the common fixed point of the mappings A, B, S and T can be easily verified by using the inequality (3.1.2) In fact, if w∈X be another common fixed point of A, B, S and T. we have

$$\max\{p^2(Sz,Tw), p^2(Tw,Sz)\}$$

$$\le ø\{p(Az, Sz)p(Bw, Tw), p(Az, Tw)p(Bw, Sz), p(Az, Sz)p(Az, Tw), p(Bw, Sz)p(Bw, Tw), p(Bw, Sw)p(Az, Sz), p(Bw, Tw)p(Az, Tw)\};$$

Thus z = w.

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