Moving Average Control Charts for Process Dispersion

V. B. Ghute and S. V. Rajmanya

Abstract - **The Shewhart type control charts based on Downton estimator (D chart) and Gini's mean difference (G chart) have been introduced in the literature for monitoring the changes in a process dispersion. Both of these charts are found to be more efficient than R chart and are very close competitor to the S chart. The aim of this paper is to develop the moving average control charts which improve the performance of the D and G charts. The proposed charts are developed for monitoring the shifts in the process variability. The average run length performance of these charts is investigated using simulation study and is compared with the originally proposed D and G charts and some other procedures proposed in the literature. The proposed charts are found to be efficient for monitoring process variability.**

Index Terms- **Control chart, Process variability, Average run length, Moving average.**

I. INTRODUCTION

Shewhart R and S control charts are most widely used to monitor process variability. The R chart is based on the sample range (R) where as S chart is based on sample standard deviation (S). Both R and S charts are easy to implement and are effective in the detection of large shifts in process standard deviation but become less effective for small shifts because they are based on only the most recent observation.

Some other control chart procedures are also available in the quality control literature to monitor process variability based on different estimates of process standard deviation. Abbasi and Miller [1] proposed the control chart based on Downton's estimate (D chart) of process standard deviation as an efficient alternative to S chart for monitoring process variability. They showed that for normally distributed process, the D chart is equally efficient to the S chart for detecting shifts in the process variability and has better performance than the R chart. Riaz and Saghir [2] proposed a control chart based on Gini's mean difference for monitoring the changes in process variability. Riaz and Saghir [3] also developed a Shewhart type control chart based on mean deviation (MD chart) to monitor process variability. They showed that MD chart is superior to R chart and is close competitor of the S chart in terms of its power for detecting shifts in process variability. Abbasi and Miller [4] proposed the EWMA chart for monitoring the changes in the process dispersion based on estimating the process standard deviation using average absolute deviation taken from the sample median.

Recently, several alternatives have been proposed in order to improve the performance of the control charts. One recent alternative developed in literature to improve the performance of Shewhart type control chart is to combine the chart with some other chart leading to a synthetic control chart. The development of the synthetic control chart for monitoring univariate and multivariate processes has been also documented by many researchers. In order improve performance of D chart, Rajmanya and Ghute [5] developed synthetic D chart as an alternative to D chart for monitoring process variability. In case of multivariate process monitoring Ghute and Shirke [6] combined the Hotelling's T^2 and CRL charts to form the multivariate synthetic T^2 chart. It was shown that the synthetic T^2 chart increases the sensitivity of the Hotelling's T^2 chart in detecting shifts in the mean vector. Ghute and Shirke [7] also developed the multivariate synthetic control chart for process dispersion by combining the generalized sample variance |S| chart and the CRL chart. The joint monitoring of mean and variability of a multivariate process using synthetic control chart is studied by Ghute and Shirke [8].

It is well known that the Shewhart-type control charts are relatively inefficient in detecting small shifts of the process parameters. Memory based control charts such as cumulative sum (CUSUM), exponentially weighted moving average (EWMA) and moving average (MA) are developed as alternative to the Shewhart charts for the detection of small process shifts in the univariate process control. They use the additional information from recent history of process hence are more effective than a Shewhart control chart in detecting small process shifts. Khoo [9] proposed Poisson moving average control chart for the number of nonconformities as an alternative to the standard c chart. Khoo [10] also suggested the design of MA chart for fraction nonconforming as an alternative to the standard p chart. Wong *et al.* [11] developed simple procedures for the design of an individual MA chart and a combined MA-Shewhart scheme. Khoo and Yap [12] proposed the use of single MA chart for joint monitoring of the process mean and variance by combining $\overline{X}$ and S charts into a single chart. Ghute and

V. B. Ghute, *Department of Statistics, Solapur University, Solapur -413255(M.S.), India. +91 9881372729*

S. V. Rajmanya, *Department of Statistics, Sangameshwar College, Solapur,(M.S.), India.*

ISSN: 2278 – 7798

Shirke [13] developed moving average control chart for monitoring mean vector of multivariate process. Ghute and Shirke [14] also developed multivariate moving average control chart for monitoring dispersion matrix of a multivariate process.

The purpose of the present paper is to develop a moving average control charts for detecting small changes produced in the process dispersion. The proposed moving average charts are based on Downton's D statistic (denoted by MA-D chart) and Gini's mean difference G statistic (denoted by MA-G chart). Simulation study is conducted to study the ARL behavior of the proposed charts. It is found that the MA-D and MA-G chart performs better than the originally proposed D chart and G chart in the detection of small to moderate shifts in the process variability. A comparative study with regard to ARL for these charts and synthetic versions of these charts is also made.

II. SHEWHART TYPE CONTROL CHARTS FOR PROCESS DISPERSION

Let $X_1, X_2, ..., X_n$ represents a random sample of size n from a normally distributed process with process mean μ and standard deviation σ and $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denote the corresponding order statistics. Downton's estimator of process standard deviation σ for normally distributed quality characteristics is given by the following statistic,

$$D = \frac{2\sqrt{\pi}}{n(n-1)} \sum_{i=1}^{n} \left(i - \frac{1}{2}(n+1) \right) X_{(i)} \qquad (1)$$

Gini's mean difference is defined as an index of variability in a population consisting of $X_1, X_2, ..., X_n$ given by following statistic,

$$G = \frac{2}{n(n-1)} \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{i=1}^{n} | X_i - X_j | \qquad (2)$$

The general structure to construct Shewhart type variability chart based on D and G statistics is presented here. Suppose T represents a dispersion statistic computed from a subgroup of size n obtained from a process that has been scaled to estimate σ (T can be any of D or G statistic). Suppose the relationship between T and σ be defined by a random variable Z as

$$Z = \frac{T}{\sigma} \qquad (3)$$

As the distribution of T is not symmetric for small to moderate sample size, Abbasi and Miller [1] used the probability limits instead of three sigma limits for the construction of T chart. They computed the probability limits by using the quantile points of the distribution of Z.

Consider a process where quality characteristic of interest X is normally distributed with mean μ and standard deviation σ. Let μ_0 and σ_0 be the in-control values of μ and σ respectively. When a shift in process standard deviation occurs, we have change from the in-control value σ_0 to the out-of-control value $\sigma_1 = \delta\sigma_0$ $(0 < \delta \neq 1)$. When $\delta = 1$, the process is considered to be in-control . For $\delta > 1$ an increase in σ occurs, process is considered to be out-of-control and an upper control limit $k^+\sigma_0$ of T chart is required, and a signal is issued if $T > k^+\sigma_0$. For $\delta < 1$ decrease in σ occurs, process is considered to be out-of-control and lower limit $k^-\sigma_0$ of T chart is required, and a signal is issued if $T < k^-\sigma_0$. The average run length (ARL) which denotes the average number of T samples required to detect a change in σ of the T chart can be calculated as,

$$ARL_T = \frac{1}{P} \qquad (4)$$

For increase in process standard deviation that is for $\delta > 1$,

$$P = \Pr(T > k^+\sigma_0 \mid \sigma = \delta\ \sigma_0) = \Pr\left(Z > \frac{k^+}{\delta} \right) = 1 - F\left(\frac{k^+}{\delta} \right) \qquad (5)$$

$$ARL_T(\delta) = \frac{1}{1 - F\left(\frac{k^+}{\delta} \right)} \qquad (6)$$

For decrease in process standard deviation that is for $\delta < 1$,

$$P = \Pr(T < k^-\sigma_0 \mid \sigma = \delta\ \sigma_0) = \Pr\left(Z < \frac{k^-}{\delta} \right) = F\left(\frac{k^-}{\delta} \right) \qquad (7)$$

$$ARL_T(\delta) = \frac{1}{F\left(\frac{k^-}{\delta} \right)} \qquad (8)$$

where $F(.)$ denotes the cumulative distribution function.

Shewhart type control chart based on D statistic can be formed by replacing T in Eq.(3) by statistic D and Shewhart type control chart based on G statistic can be formed by replacing T in Eq.(3) by statistic G.

III. MOVING AVERAGE CONTROL CHARTS FOR PROCESS DISPERSION

This section discusses the construction of the moving average control chart which is based on computing the moving averages of T statistics (denoted MA-T chart). The moving average statistic of span *w* at time *i* for a sequence of T statistics computed as

$$MT_i = \frac{T_i + T_{i-1} \ldots + T_{i-w+1}}{w}, \text{ for } i \geq w \quad (9)$$

For periods $i < w$, we compute the average of available charting statistic. In other words, average of all T observations up to period i defines moving average. The MA-T chart is constructed by plotting the MT_i statistics on the chart against the sample number *i*. An out-of-control signal is issued when MT_i is smaller than the lower control limit (*LCL*) or larger than the upper control limit (*UCL*).

Knowledge of the statistical distribution of the control chart statistic is needed to calculate the control limits of the chart. If the exact distribution of a control chart statistic is unknown or intractable, then control limits can be calculated from either an approximate distribution or from a Monte Carlo simulation.

The exact distribution of control chart statistics $\mathrm{MT_i}$ is unknown. Control chart statistic MT_i, $i = 1, 2, \ldots$ is a sequence of dependent variables, it may not easy to obtain the exact distribution of these statistics. Therefore, simulation technique is used to obtain control limits and ARL values. Since T is the sequences of positive values, the value of charting statistics MT_i never negative. Therefore lower control limit of the proposed control charts is taken as zero and upper control limit is obtained by simulation so that the chart has the desired in-control ARL value. For the moving average control charts, the control limits for period $i < w$ are wider than that of $i \geq w$. However, here we used a constant upper control limit since *w* is usually a small value such as 2, 3, 4 or 5. The proposed charts are constructed by plotting the corresponding moving average statistics on the chart with the simulated upper control limit.

The moving average control statistic based on D statistic can be formed by replacing T in Eq.(9) by statistic D . Thus moving average control chart statistic based on D statistic is given by

$$MD_i = \frac{D_i + D_{i-1} \ldots + D_{i-w+1}}{w}, \text{ for } i \geq w \quad (10)$$

the chart based on this statistic is denoted by MA-D chart.

Similarly, he moving average control statistic based on Gini G statistic can be formed by replacing T in Eq.(9) by statistic G . Thus moving average control chart statistic based on G statistic is given by

$$MG_i = \frac{G_i + G_{i-1} \ldots + G_{i-w+1}}{w}, \text{ for } i \geq w \quad (11)$$

the chart based on this statistic is denoted by MA-G chart.

IV. PERFORMANCE EVALUATIONS OF PROPOSED CHARTS

To assess the performance of the proposed moving average control charts, the ARL is used as performance measure. Simulation study based on three subgroups each of size $n = 5, 8$ and 10 is used to determine the ARL of in-control and out-of-control processes. Let us assume that the in-control process is a normally distributed with the mean μ_0 and variance σ_0^2 . The out-of-control process is a normally distributed with the same mean and changed variance σ_1^2 . The amount of shift in the process variability is given by $\delta = \frac{\sigma_1}{\sigma_0}$. Further it is assumed that μ_0 is zero and $\sigma_0^2 = 1$. Using the simulation, the upper control limit (*UCL*) of the proposed moving average charts is obtained for $w = 2$, 3, 4 and 5 for all subgroup sizes, *n*, so that in-control ARL of the chart is approximately 200. The out-of-control ARL values of the control charts for various shifts in the process standard deviation are then computed using 10000 simulations.

To compare the performance of the proposed moving average charts with other control charts, each chart is designed so that in-control ARL is approximately 200.

Tables 1-3 display the ARL comparison of D chart, synthetic D chart and MA-D chart for various shifts of magnitude δ in the process standard deviation with sample size $n = 5$, 8 and 10 respectively.

From Tables 1-3, we observe that for any range of shifts, the MA-D chart consistently produces smaller out of control ARL than the Shewhart type D chart. ARL comparison of the MA-D chart with synthetic D chart show that for $w \geq 4$, the MA-D chart produces smaller out of control ARL than the synthetic D chart.

Tables 4-6 display the ARL comparison of G chart, synthetic G chart and MA-G chart for various shifts of magnitude δ in the process standard deviation with sample size $n = 5$, 8 and 10 respectively.

From Tables 4-6, we observe that for any range of shifts, the MA-G chart consistently produces smaller out of control ARL than the Shewhart type G chart. ARL comparison of the MA-G chart with synthetic G chart show that for $w \geq 4$, the MA-G chart produces smaller out of control ARL than the synthetic G chart.

V. CONCLUSIONS

In this paper, two moving average control charts based on D and G statistics are developed for monitoring process variability. The proposed MA-D and MA-G charts produce a significant ARL improvement in comparison with the Shewhart type and synthetic control charts. As the moving average control charts has a better performance than the traditional charts, it is recommended to use the proposed control charts for monitoring small shifts in process variability.

ISSN: 2278 – 7798

Table 1 ARL Comparisons for $n = 5$.

Shift δ	MA-D chart				D chart $k = 2.068$	Synthetic D Chart L=17 k =1.445
	$w = 2$ UCL= 1.723	$w = 3$ UCL= 1.573	$w = 4$ UCL= 1.484	$w = 5$ UCL= 1.425		
1.0	200.12	200.48	201.53	199.05	200	200
1.1	55.65	48.65	43.32	40.38	66.65	43.92
1.2	21.46	18.41	15.78	13.91	29.67	15.89
1.3	11.10	9.13	7.74	6.85	15.56	8.34
1.4	6.90	5.65	4.83	4.35	9.59	5.38
1.5	4.73	3.76	3.36	3.04	6.49	3.92
2.0	1.84	1.61	1.49	1.42	2.31	1.79
2.5	1.31	1.22	1.18	1.15	1.54	1.33
3.0	1.15	1.11	1.08	1.07	1.26	1.17

Table 2 ARL Comparisons for $n = 8$

Shift δ	MA-D chart				D chart k =1.7763	Synthetic D Chart L=12 k =1.595
	$w = 2$ UCL= 1.530	$w = 3$ UCL= 1.4222	$w = 4$ UCL= 1.358	$w = 5$ UCL= 1.314		
1.0	199.12	200.54	198.86	201	200	200
1.1	41.72	35.52	30.38	27.98	54.0	32.80
1.2	14.69	11.53	9.77	8.53	20.62	10.75
1.3	6.80	5.35	4.76	4.06	10.40	5.27
1.4	4.13	3.35	2.91	2.58	6.05	2.49
1.5	2.88	2.36	2.10	1.92	4.13	2.56
2.0	1.31	1.20	1.15	1.13	1.55	1.31
2.5	1.09	1.05	1.04	1.03	1.17	1.08
3.0	1.03	1.02	1.01	1.01	1.07	1.03

Table 3 ARL Comparisons for $n = 10$.

Shift δ	MA-D chart				D chart $k = 1.676$	Synthetic D Chart L=12 k =1.5192
	$w = 2$ UCL= 1.462	$w = 3$ UCL= 1.368	$w = 4$ UCL= 1.3128	$w = 5$ UCL= 1.2744		
1.0	200.25	200.58	201.28	200.89	200	201
1.1	35.71	29.39	26.67	23.01	48.33	28.61
1.2	11.87	9.21	7.78	6.52	17.45	8.66
1.3	5.59	4.24	3.62	3.34	8.27	4.41
1.4	3.30	2.68	2.34	2.09	4.86	2.89
1.5	2.34	1.93	1.72	1.59	3.28	2.16
2.0	1.18	1.11	1.09	1.07	1.34	1.18
2.5	1.03	1.02	1.02	1.01	1.09	1.05
3.0	1.01	1.01	1.00	1.00	1.02	1.01

Table 4 ARL Comparisons for n = 5.

Shift δ	MA-G chart				G chart k =2.3342	Synthetic G Chart L=17 k =2.0788
	w = 2 UCL= 1.990	w = 3 UCL= 1.814	w = 4 UCL= 1.713	w = 5 UCL= 1.6442		
1.0	200.6	199.89	200.78	201.07	200	200
1.1	54.88	47.81	43.65	39.62	65.94	43.56
1.2	21.61	17.73	15.93	13.81	29.02	16.12
1.3	11.06	9.03	7.81	7.06	15.63	8.25
1.4	6.70	5.52	4.83	4.18	9.67	5.33
1.5	4.69	3.87	3.29	2.95	6.52	3.96
2.0	1.84	1.58	1.49	1.40	2.33	1.80
2.5	1.31	1.22	1.18	1.14	1.53	1.33
3.0	1.15	1.11	1.08	1.08	1.26	1.17

Table 5 ARL Comparisons for n = 8

Shift δ	MA-G chart				G chart k = 2.005	Synthetic G Chart L= 12 k = 1.7997
	w = 2 UCL= 1.767	w = 3 UCL= 1.6409	w = 4 UCL= 1.5677	w = 5 UCL= 1.517		
1.0	201.86	199.36	200.87	200.29	200	200
1.1	41.11	34.85	30.60	26.69	53.86	32.97
1.2	14.20	11.44	9.81	8.56	20.71	10.87
1.3	6.91	5.57	4.67	4.11	10.30	5.28
1.4	4.16	3.35	2.85	2.58	6.10	3.47
1.5	2.85	2.34	2.11	1.89	4.08	2.52
2.0	1.30	1.20	1.16	1.13	1.56	1.31
2.5	1.08	1.05	1.04	1.04	1.17	1.09
3.0	1.03	1.02	1.01	1.01	1.06	1.03

Table 6 ARL Comparisons for n = 10.

Shift δ	MA-G chart				G chart k = 1.8902	Synthetic G Chart L= 12 k = 1.713
	w = 2 UCL= 1.689	w = 3 UCL= 1.580	w = 4 UCL= 1.515	w = 5 UCL= 1.472		
1.0	201.08	201.97	201.28	200.89	200	200
1.1	36.30	29.26	26.67	23.01	48.34	28.68
1.2	11.59	9.04	7.78	6.52	17.25	8.69
1.3	5.55	4.39	3.62	3.34	8.41	4.45
1.4	3.31	2.68	2.34	2.09	4.88	2.87
1.5	2.33	1.94	1.72	1.59	3.27	2.17
2.0	1.18	1.11	1.09	1.07	1.35	1.19
2.5	1.04	1.02	1.02	1.01	1.08	1.04
3.0	1.01	1.01	1.00	1.00	1.02	1.01

ISSN: 2278 – 7798

REFERENCES

[1] S.A. Abbasi and a. Miller, "D chart: An efficient alternative to monitor process dispersion," *Proceedings of the World Congress on Engineering and Computer Science* (WCECS 2011); 2, *San Francisco, USA*, 2011.

[2] M. Riaz and A. Saghir, "Monitoring process variability using Gini's mean difference," *Quality Technology and Quantitative Management,* vol. 4, pp.439-454, 2007.

[3] M. Riaz and A. Saghir, "A mean deviation-based approach to monitor process variability," *Journal of Statistical Computation and Simulation,* vol. 79, pp. 1173-1193, 2009.

[4] S. A. Abbasi and A. Miller, "MDEWMA: an efficient and robust alternative to monitor process dispersion," *Journal of Statistical Computation and Simulation,* vol. 1, pp. 1-22, 2011.

[5] S. V. Rajmanya and V. B. Ghute, "A synthetic control chart for monitoring process variability," *Quality and Reliability Engineering International*; Published Online, DOI: 10.1002/qre.1551, 2013.

[6] V. B. Ghute. and D. T. Shirke, "A multivariate synthetic control chart for monitoring process mean vector," *Communications in Statistics- Theory and Methods,* vol. 37: pp. 2136-2148, 2008.

[7] V. B. Ghute. and D. T. Shirke, "A multivariate synthetic control chart for process dispersion," *Quality Technology and Quantitative Management,* vol. 5, pp. 271-288, 2008.

[8] V. B. Ghute. and D. T. Shirke, "Joint monitoring of multivariate process using synthetic control charts," *International Journal of Statistics and Management Systems,* Vol.2, pp.129-141, 2007.

[9] M. Khoo, "Poisson moving average versus c chart for nonconformities," *Quality Engineering,* vol.16, pp.525-534, 2004.

[10] M. Khoo, "A moving average control chart for monitoring the fraction non-conforming," *Quality and Reliability Engineering International*. vol. 20, pp. 617-635, 2004.

[11] H. Wong, F. Gan, F. and T. Chang, "Design of moving average control chart," *Journal of Statistical Computation and Simulation.* vol. 74, pp. 47-62, 2005.

[12] M. Khoo and P. Yap, "Joint monitoring of process mean and variability with a single moving average control chart," *Quality Engineering*. vol. 17, pp. 51-65, 2005.

[13] V. B. Ghute and D. T. Shirke, **"**A multivariate moving average control chart for mean vector," *Journal of Academia and Industrial Research*, vol. 1(12), pp. 795-800, 2013.

[14] V. B. Ghute and D. T. Shirke, **"**Multivariate moving average control chart for process variability," *International Journal of Statistics and Analysis*, vol. 3, pp. 97-104, 2013.

V. B. Ghute is an Associate Professor and Head at the Department of Statistics, Solapur University, Solapur, (MS), India. He received his M.Sc. and Ph.D. degrees in Statistics from Shivaji University, Kolhapur, India. He is a life member of International Indian Statistical Association. He is also a life member of Indian Society for Probability and Statistics and Shivaji University Statistics Teachers' Association. His current research includes statistical process control, statistical computing, and statistical inference.

S. V. Rajmanya is an Associate Professor in the Department of Statistics, Sangameshwar College, Solapur, (MS), India. She received her M.Sc. degree in Statistics from University of Pune, Pune, India. Presently she is a Ph.D. student in the Department of Statistics, Solapur University, Solapur, India. She is a life member of Indian Society for Probability and Statistics.