MMSE Denoising of 2-D Signals Using Consistent Cycle Spinning Algorithm

Bodduluri Asha, B. Leela kumari

Abstract: It is well known that in a real world signals do not exist without noise, which may be negligible under certain conditions. The process of noise removal is generally known as signal denoising .In this project a technique for the implementation of MMSE (Minimum Mean Square Error) for the signal denoising using CCS algorithm is implemented. This method exploits the link between the discrete gradient and Haar-wavelet shrinkage. The Additive White Gaussian Noise (AWGN) involved in the 2-D image can be removed using various methods and compare those results obtained. The performance of denoising algorithm is usually taken to be mean square error (MSE) based, between the original signal and its reconstructed version.

Keywords: Signal Denoising, MMSE, Wavelet Domain, Statistical Estimation.

I. INTRODUCTION

In this paper, we consider the Bayesian perspective and proposed a new algorithm for efficiently computing MMSE-TV Denoising[1]. This approach is based on cycle spinning and it exploits the link between the discrete gradient and Haar-wavelet shrinkage. It consists in an adaption of cycle spinning algorithm[8] which was successfully used to compute MAP estimators for sparse stochastic signals. In substance, we extend the original CCS algorithm by replacing MAP shrinkage functions with their MMSE counterparts. Cycle spinning implies a redundant representation. Thus, not every set of wavelet-domain coefficients can be perfectly inverted back to the signal domain. Problems are faced if the estimated coefficients violate the invariability condition. It can be resolved through consistency, which refers to a technique whereby the solutions obtained by CCS are made perfectly invertible. This is achieved by restricting the wavelet-domain solution to the space spanned by the basis function of the transform, which results in a norm equivalence between signal domain and wavelet-domain estimations.

The main advantage of CCS is its computational simplicity, and it reduces the difficult estimation problem to the iterative application of a scalar shrinkage function in the shift-invariant Haar wavelet-domain. This method can be applicable to denoising of 1-D signals and also to 2-D images[7]. This can also be extended to 3-D signals and video communication. For the objective evaluation of the algorithm, we perform estimation of Levy processes[9], [10] that have independent and stationary increments. Recently, we have developed a message passing algorithm for computing the MMSE estimator for Levy processes[4]. Experimental results show that CCS performs as well as the message passing algorithm [4] running substantially faster.

II. SIGNAL DENOISING

Signal denoising background is mainly comprises of two sections as mentioned below such as Wavelet based Denoising using transforms and Total Variation Regulation.

i) Wavelet based Denoising

Transform based approaches plays a prominent rule in signal enhancement, signal denoising, etc. Wavelet based denoising is a novel approach to remove unwanted artifacts from the respective signal to get the normal form of signal from the respective signal to get the normal form of signal from the abnormal form[7]. Wavelet based denoising approach is performed in three steps.

A) Perform the wavelet based approach of the data u = Wy € R^m, where m≥n. when m = n then the respective transform approach is orthogonal.

B) Later the denoised algorithm is reresented by f_w: Rm → Rm that maps data to be estimated by w = $W^T\acute{w}$ € R^N

ISSN: 2278 – 7798

C) Finally the inverse transform is applied in order to return to the image domain to obtain ˆ x = W^{T} w ϵ R^N

In the sequel we restrict our attention to tight wavelet frames, which implies that the transform preserves inner-products and is self-reversible.

$$\forall x \in \mathbb{R}^n, W^T W x = x.$$

This also implies that the transform is norm-preserving, with $||\boldsymbol{x}||_2^2 = ||\boldsymbol{W_x}||_2^2$. Traditionally, the estimation problem in the wavelet domain is expressed as

$$\widehat{w} = f_W(u) = \underset{x \epsilon \mathbb{R}^n}{argmin} \left\{ \frac{1}{2} ||w - u||_2^2 + \Phi(\mathrm{x}) \right\}$$

The least-squares term in the above equation is justified by the fact that the transform W is norm preserving. Nevertheless, we show in the sequel that equation is suboptimal when W is redundant.

The powerful aspect of wavelet decomposition is that it has a decorrelating effect on many natural signals. Thus, it is very common to select the regularize $\emptyset_{\mathbf{w}}$ to be separable. For example, the popular wavelet soft thresholding algorithm[11] is obtained by using the non-smooth convex function $\emptyset_w$ (.)= $||.||_1$ that favors sparse wavelet-domain solutions and admits a closed form solution $\widehat{\boldsymbol{w}} = \boldsymbol{\eta}(\boldsymbol{u}:\boldsymbol{\lambda})$. The soft-thresholding function $\boldsymbol{\eta}$ is applied component-wise on the detail wavelet coefficients.

From a Bayesian MAP perspective, the soft-Thresholding algorithm inherently assumes the detail wavelet coefficients to be i.i.d. Laplace random variables. Alternatively, one could assign other i.i.d. priors $\boldsymbol{p_w}$ to the wavelet coefficients and identify a corresponding MAP proximity operator $\boldsymbol{\eta_{MAP}}$ [12]. Better yet, one can choose the pointwise MMSE estimator for specific priors [13], [14]. A convenient way of determining the optimal shrinkage function in the AWGN scenario is to rely on Stein's formula[6], [15]

$$\begin{aligned} \widehat{w}_i = \eta_{MMSE}(u_i) &= \mathbb{E}[\omega | u_i] \\ &= u_i + \sigma_i^2 \frac{d}{du} \log\, p_u(u_i) \end{aligned}$$

Where σ_i^2 is the variance of the noise of the i th wavelet-domain coefficient $\boldsymbol{p_u} = \boldsymbol{p_w} * \boldsymbol{g_{\sigma i}}$ and is the probability distribution of the noisy measurement with the noise distribution

$$g_\sigma(w) = \frac{1}{\sqrt[\sigma]{2\pi}} e^{-w^2/_{2\sigma^2}}$$

A major drawback of the wavelet-based approach is that MMSE estimation in the wavelet domain is not in general equivalent to MMSE estimation in the signal domain. The exception happens only when the wavelet transform is orthogonal. However, empirical results reported over the years show that wavelet-based algorithms using over complete transforms are more effective than those using orthogonal transforms.

ii) Total-Variation Regularization

TV denoising operates directly in the signal domain by solving the optimization problem with the regularize $\boldsymbol{\emptyset(x) = \lambda \| D(x) \|_1}$, where $\boldsymbol{\lambda} > 0$ is the regularization parameter $\boldsymbol{D(x) \epsilon R^n}$ and is a vector that contains the discrete gradient of at all positions, defined component-wise as $[D(x)]_i = x_{i+1} - x_i$, $\forall i \in [1 \dots . n-1]$
Where we assume that $x_0 = 0$.

III.CONSISTENT CYCLE SPINNING

In this section, we present the CCS algorithm, which is in some sense an extension to both conventional wavelet denoising and TV regularization with regularized least-squares data terms.

A. Formulation of the Problem

As mentioned in above section, the fact that the frame is tight implies that always holds. However, since $\boldsymbol{m > n}$, there are some $w \in \mathrm{R}^n$ such that the converse equality $\boldsymbol{WW^Tw = w}$, does not hold. This in turn implies that the solution obtained by solving the equation in the tight wavelet-frame domain is not consistent, in the sense that the wavelet transform of the final solution $W^T\widehat{w} = \hat{x}$ is not necessarily equaled to $\widehat{w}$ obtain. However, since the desired solution always lives in the signal domain, it makes sense to constraint the wavelet-domain estimate to the consistent subspace, which also makes it perfectly invertible. Such estimation can be performed by solving the following constrained optimization problem

$$\widehat{w} = \mathrm{argmin}\, \mathcal{J}(u, w) \qquad s.t \qquad WW^Tw = w,$$

With $\quad \mathcal{J}(u, w) = \frac{1}{2} ||w - u||_2^2 + \Phi(\mathrm{x})$

The estimation problem combines better approximation capabilities of redundant representations with wavelet-domain solutions that behave as if the transform were truly orthogonal.

ISSN: 2278 – 7798

Reconstruction with undecimated wavelet transforms is commonly referred to as cycle spinning[8]. If we consider an undecimated single-level Haar expansion, which involves sums and differences[16], then it is clear that each element of the discrete gradient can be uniquely mapped to the detail coefficients of $\boldsymbol{w} = \boldsymbol{W}\boldsymbol{x}$ by replacing each finite difference by some corresponding Haar coefficient. Let $\boldsymbol{w}^{a}$ and $\boldsymbol{w}^{d}$ denote the approximation and detail coefficients in , respectively. Then, for the case of $\emptyset_{\boldsymbol{w}}(\boldsymbol{w}) = \boldsymbol{\lambda}\sqrt{\boldsymbol{2}}\ \|\boldsymbol{w}^{\boldsymbol{d}}\|$, the optimization in above equation exactly performs TV denoising[7]. The optimization also admits a simple interpretation as estimation of a sparse and consistent wavelet-domain solution.

B. Iterative Consistent MMSE Shrinkage

We now consider the MMSE version of TV denoising where we set the regularizer to

$$\Phi_W(\mathrm{w}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \emptyset_{\mathrm{MMSE}}\,(\mathrm{w}_{\mathrm{i}}^{\mathrm{d}})$$

and where the potential function $\emptyset_{\mathrm{MMSE}}$ corresponds to MMSE shrinkage. Observe that, when $\emptyset_{\mathrm{MMSE}}\,(.) = |.|$, the problem reduces to TV denoising. In general, $\emptyset_{\mathrm{MMSE}}$ does not necessarily admit an analytic formula. However, one can formally characterize it as [6]

$$\emptyset_{\mathrm{MMSE}}\,(\mathrm{w}\,) = -\frac{1}{2}(\eta_{\mathrm{MMSE}}^{-1}\,(\mathrm{w}) - \mathrm{w})^2 - \log\ p_u(\eta_{\mathrm{MMSE}}^{-1}\,(\mathrm{w}))$$

for all w in the image of η_{MMSE} , where $\eta_{\mathrm{MMSE}}^{-1}$ denotes the inverse of the MMSE shrinkage. A practical optimization scheme for the constrained minimization problem with potential function $\emptyset_{\mathrm{MMSE}}$ can be obtained by using augmented-Lagrangian approach[17], which casts the constrained problem as a sequence of unconstrained problems. The idea is to replace the objective function $\mathcal{T}$ with the new penalty function

$$\mathcal{L}(w,x) = \mathcal{J}(u,x) + \frac{\mathcal{T}}{2}||w - Wx||_2^2 - \mu^T(w - Wx)$$

Where $\boldsymbol{\mathcal{T}} > \boldsymbol{0}$ is the penalty parameter and $\boldsymbol{\mu} \in \mathbb{R}^{\boldsymbol{m}}$ is the vector of Lagrange multipliers. The condition $\boldsymbol{w} = \boldsymbol{W}\boldsymbol{x}$ asserted by the penalty function constraints w to the column space of W, which is equivalent to the consistency condition $\boldsymbol{w} = \boldsymbol{W}\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{w}$. Although the quadratic penalty term in $\mathcal{T}$ does not influence the final solution, it typically improves the convergence behavior of the iterative optimization[17]. To solve the minimization of the objective, we alternate between solving the problem for with fixed $\boldsymbol{w}$ and $\boldsymbol{x}$ vice versa.

Given the noisy wavelet-domain measurements, penalty parameter $\boldsymbol{\mathcal{T}} > \boldsymbol{0}$, regularizer ϕ_w and initial signal-domain solution, CCS estimation proceeds as:

1) *Initialize*: Set t=0 and $\mu^0 = 0$.

2) *Update* $\widehat{w}$: Minimize $\boldsymbol{\mathcal{L}}$ with respect to **w** with **x** fixed

$$\widehat{w}^{t+1} = \arg\min \mathcal{L}(w, \hat{x}^t) \qquad w \in \mathbb{R}^m$$

$$= \mathrm{argmin}\left\{\frac{1}{2}||w - \tilde{u}||_2^2 + \tilde{\lambda}\phi_W(w)\right\}$$

$$= \eta_{MMSE}\,(\tilde{u})$$

where

$$\tilde{u} = {(u + \tau W\hat{x}^t + \mu^t)}\big/{(1+\tau)}\ and\ \tilde{\lambda} = 1/(1+\tau)$$

The function η_{MMSE} is applied component wise on the detail coefficients of $\widetilde{w}$. It is a 1D shrinkage operator and can be implemented as a lookup table.

3) *Update* $\tilde{x}$*:* Minimize $\boldsymbol{\mathcal{L}}$ with respect to **w** with **x** fixed

$$\hat{X}^{t+1} = \arg\min \mathcal{L}(\widehat{w}^{t+1}, X) \qquad w \in \mathbb{R}^n$$

$$= W^T(\widehat{w}^{t+1} - \frac{\mu^t}{\tau})$$

4) *Update* μ: Lagrange multipliers are updated according to the simple rule

$$\mu^{t+1} = \mu^t - \tau(\widehat{w}^{t+1} - W\hat{x}^{t+1})$$

5) Set t = t + 1 and proceed to Step 2.

For each iteration t=1,2,…. , the proposed update rules produce estimates of the true signal $\hat{x}^t$. The computational complexity of the algorithm ***O(n)*** is per iteration, since it reduces to the evaluation of wavelet transforms and point wise non-linearities.

ISSN: 2278 – 7798

SIMULATION RESULTS

For 1-D signal:

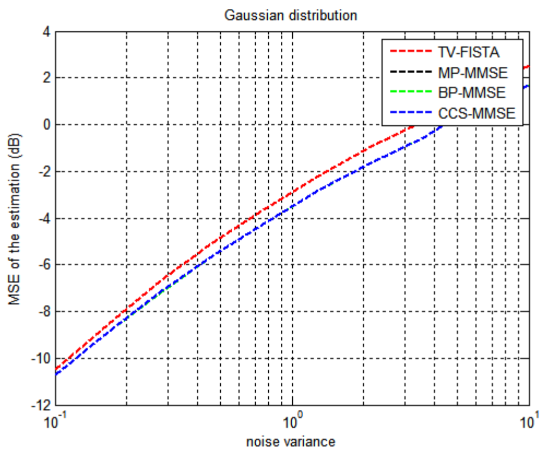

Figure 1: Comparison of various estimation methods on Lévy processes in terms of MSE and noise variance in Gaussian distribution

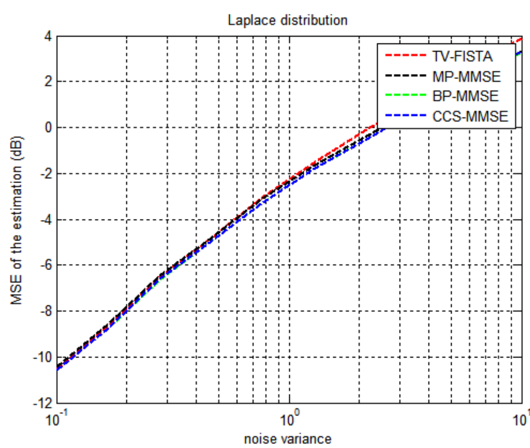

Figure 2: Comparison of various estimation methods on Lévy processes in terms of detailed MSE and noise variance in Laplace distribution

For 2-D signal:

Figure 3: Input image for the estimation of CCS algorithm

Figure 4: Output image for the estimation of CCS algorithm

Signal type	Various Algorithms			
	TV FISTA	BP MMSE	MP MMSE	CCS MMSE
1-D	1.6949 Sec	0.15546 Sec	3.6371 sec	0.00036914 Sec
2-D	2.3609 Sec	6.6253 Sec	0.21962 sec	0.00048076 Sec

Table 1: Comparison of reconstruction time for different techniques

CONCLUSION

The problem as a penalized least-squares regression in the redundant wavelet domain is proposed in this paper. The method exploits the link between the discrete gradient and Haar-wavelet shrinkage with cycle spinning. Thus by imposing additional constraints, finally our method finds the wavelet-domain solution that is mutually consistent. For example, the average reconstruction speed of MP-MMSE was 3.63 and 5.03 seconds for Gaussian and Laplace increments, respectively, whereas the average reconstruction speed of CCS MMSE was only 0.00036914 seconds for the same signals.

ISSN: 2278 – 7798

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ISSN: 2278 – 7798