# Approximation of real definite integrals via hybrid quadrature domain

**Saumya ranjan Jena, Prabhas Dash**

*Abstract*— **This paper deals with construction of mixed quadrature rule of precision seven using linear combination of Gauss-Legendre-3-point rule and Lobatto-4-point rule each of degree of precision five. Taking the convex combination of above mixed quadrature rule of precision seven with Richardson extrapolation the so called hybrid quadrature rule of degree of precision nine is formulated. The rule is numerically verified and error bound is determined.**

*Index Terms*— **Degree of precision, Hybrid quadrature rule, Mixed quadrature rule, Richardson extrapolation, Simpson's $\frac{1}{3}$ rd rule**
**MSC(2000) 65D30, 65D32.**

## I. Introduction

Real definite integral of the form

$$I(f)=\int_a^b f(x)\,dx \qquad (1)$$

can be approximated by using (i) Gauss-Legendre-3-point rule (ii) Lobatto-4-point rule by the monomial transformation $x=\frac{(b-a)t+(b+a)}{2}$ to transform $[a,b]$ to $[-1,1]$ and the convex combination of (i) and (ii) is mixed with Richardson extrapolation. The nodes of n-point Gauss-Legendre rule and n-point Lobatto rule are zeros of $P_n(x),\ n\geq 2$ and $P'_{n-1}\ ,n\geq 3$ respectively. In these three rules, as we move from lower order rule to higher order rule, almost all the information obtained in computing the former gets discarded because the nodes and weights are different for different value of $n$. Keeping these facts "[3]" in view, we desire to construct a hybrid quadrature rule of precision nine. The construction of hybrid quadrature rule is outlined in the following sections as follows. In

section II we formulate the so called hybrid quadrature rules $R_{RL4GL3}(f)$. In section III an error bound of the said rule is established and in section IV the rule is numerically verified.

***Saumya ranjan Jena,*** *Department of Mathematics, School of Applied Science, Kiit University, Bhubaneswar-751024,Odisha, India , Mob.: +91-9437127967.*
***Prabhas Dash,*** *Department of Mathematics, School of Applied Science, Kiit University, Bhubaneswar-751024,Odisha, India , Mob.: +91-9853766773.*

## II. CONSTRUCTION OF HYBRID QUADRATURE RULE OF PRECISION NINE

For the approximate evaluation of the real integral

$$I(f)=\int_{-1}^{1} f(x)\,dx \qquad (2)$$

a mixed quadrature rule of Lobatto-4 point rule and Gauss-Legendre-3-point rule is

$$I(f)=\int_{-1}^{1} f(x)\,dx \approx R_{L4GL3}(f)$$

$$=\frac{1}{7}\left[4R_{GL3}(f)+3R_{L4}(f)\right] \qquad (3)$$

The truncation error obtained in this approximation is given by,

$$E_{L4GL3}(f)=\frac{-32}{7875\times 8!}f^{viii}(0)-\frac{2560}{2402625\times 10!}f^{x}(0)-..(4)$$

$$I(f)=\int_{-1}^{1} f(x)dx \approx I_n^{(k)}=\frac{4^k I_n^{k-1}-I_{\frac{n}{2}}^{(k-1)}}{4^k-1};\quad n\geq 2^k \qquad (5)$$

and $k\geq 1$.

For $k=3$ Richardson extrapolation rule is,

$$I_n^{(3)}=\frac{64I_n^{(2)}-I_{\frac{n}{2}}^{(2)}}{63} \qquad (6)$$

For $n=8$,

$$I_8^{(3)}=\frac{64I_8^{(2)}-I_4^{(2)}}{63} \qquad (7)$$

For $n\geq 4$,

$$I_n^{(2)}=\frac{161_n^{(1)}-I_{\frac{n}{2}}^{(1)}}{15} \qquad (8)$$

For $n=8$,

$$I_8^{(2)}=\frac{161_8^{(1)}-I_4^{(1)}}{15} \qquad (9)$$



ISSN: 2278 – 7798 

using Simpson's $\frac{1}{3}$ rd rule in "(7)", "(8)" and "(9)" and substituting the obtained result in "(6)" for $h=\frac{1}{4}$ Richardson extrapolation rule $R_{RExt}(f)$ is

$$I_8^{(3)}=\frac{1}{2835}[217f_0+1024f_1+352f_2+1024f_3+436f_4+1024f_5+352f_6+1024f_7+217f_8] \quad (10)$$

each of the rule in "(3)" and "(10)" is of precision seven. Let $E_{LGL3}(f)$ and $E_{RExt}(f)$ denote the error in approximating the integral $I(f)$ by the rule "(3)" and "(10)" respectively. Then

$$I(f)=R_{LGL3}(f)+E_{LGL3}(f) \quad (11)$$

$$I(f)=R_{RExt}(f)+E_{RExt}(f) \quad (12)$$

Now assume $f(x)$ to be sufficiently differentiable in $-1\le x\le 1$, using Maclaurin's theorem, "(2)" can be written as,

$$I(f)=2f(0)+\frac{2}{3!}f^{ii}(0)+\frac{2}{5!}f^{iv}(0)+\frac{2}{7!}f^{vi}(0)+\frac{2}{9!}f^{viii}(0)+\frac{2}{11!}f^{x}(0)+\dots\dots\dots \quad (13)$$

Substituting

$f_0=f(-1)$ , $f_1=f\left(\frac{-3}{4}\right)$ , $f_2=f\left(\frac{-2}{4}\right)$ , $f_3=f\left(\frac{-1}{4}\right)$ , $f_4=f(0)$ , $f_5=f\left(\frac{1}{4}\right)$ , $f_6=f\left(\frac{2}{4}\right), f_7=f\left(\frac{3}{4}\right), f_8=f(1)$

in "(10)", we have

$$R_{RExt}(f)=2f(0)+\frac{2}{3!}f^{ii}(0)+\frac{2}{5!}f^{iv}(0)+\frac{2}{7!}f^{vi}(0)+\frac{163}{80\times 9!}f^{viii}(0)+\frac{1639}{768\times 11!}f^{x}(0)+\dots \quad (14)$$

The error in Richardson extrapolation quadrature formula is

$$E_{RExt}(f)=\int_{-1}^{1}f(x)\,dx-R_{RExt}(f) \quad (15)$$

$$E_{RExt}(f)=\frac{-1}{240\times 8!}f^{viii}(0)-\frac{103}{8448\times 10!}f^{x}(0)+\dots\dots \quad (16)$$

As the error contains at least eighth derivative of the integrand functions, the degree of precision is seven.

Now multiplying "(11)" by $\left(\frac{1}{16}\right)$ and "(12)" by $\left(\frac{-32}{525}\right)$ and adding the results we get,

$$I(f)=\frac{525}{13}R_{L4GL3}(f)-\frac{512}{13}R_{RExt}(f)+\frac{2}{975\times 10!}f^{x}(0)+\dots\dots \quad (17)$$

$$I(f)=R_{RL4GL3}(f)+E_{RL4GL3}(f) \quad (18)$$

Where

$$R_{RL4GL3}(f)=\frac{1}{13}[525R_{L4GL3}(f)-512R_{RExt}(f)] \quad (19)$$

The notation $R_{RL4GL3}(f)$ and $E_{RL4GL3}(f)$ are mixed quadrature rule and its error obtained by Gauss-Legendre-3 point rule and Labatto-4 point quadrature rule with a quadrature obtained from Richardson extrapolation. The truncation error generated in this approximation is given by,

$$E_{RL4GL3}(f)=\frac{1}{13}[525E_{L4GL3}(f)-512E_{RExt}(f)] \quad (20)$$

$$E_{RL4GL3}(f)=\frac{2}{975\times 10!}f^{10}(0)+\dots\dots \quad (21)$$

In this hybrid quadrature rule the error consists at least tenth order derivatives. Thus this hybrid quadrature rule is theoretically capable of computing exactly all polynomials of degree up to nine. Thus its degree of precision is nine. Theoretically this formula is greater than the constituent formulae. The "(19)" may be called hybrid quadrature rule as it is constructed from two different type rules of the same precision.

## III. ERROR ANALYSIS OF HYBRID QUADRATURE RULE

An asymptotic error estimate and an error bound of "(19)" given by the following theorems 1 and theorem 2 respectively.

**Theorem.1**
Let $f(x)$ be sufficiently differentiable function in $[-1,1]$ then the error $E_{RL4GL3}(f)$ associated with the rule $R_{RL4GL3}(f)$ is given by,

$$|E_{RL4GL3}|\approx\frac{2}{975\times 10!}\left|f^{x}(0)\right| \quad (22)$$

The proof follows from "(21)"

**Theorem.2**
The bounds of the truncation error $E_{RL4GL3}(f)=I(f)-R_{RL4GL3}(f)$



ISSN: 2278 – 7798 

is given by,

$$E_{RL4GL3}(f) \le \frac{4M}{7875 \times 8!} \quad (23)$$

Where,

$$M = \max_{-1 \le x \le 1} \left| f^{ix}(x) \right| \quad (24)$$

**Proof:**

We have,

$$E_{L4GL3}(f) = \frac{-32}{7875 \times 8!} f^{viii}(\eta_1); \quad \eta_1 \in [-1, 1] \quad (25)$$

And

$$E_{RExt}(f) = \frac{-1}{240 \times 8!} f^{viii}(\eta_2), \quad \eta_2 \in [-1, 1] \quad (26)$$

From "[1]" and " (20)" let

$$K = \max_{-1 \le x \le 1} \left| f^{viii}(x) \right| \quad (27)$$

$$k = \min_{-1 \le x \le 1} \left| f^{viii}(x) \right| \quad (28)$$

As $f^{viii}(x)$ is continuous and $[-1, 1]$ is compact, hence there exist points $b$ and $a$ in the interval $[-1, 1]$ such that $K = f^{viii}(b)$ and $k = f^{viii}(a)$

Thus,

$$E_{RL4GL3}(f) \le \frac{2}{7875 \times 8!}[f^{viii}(b) - f^{viii}(a)]$$

$$= \frac{2}{7875 \times 8!}(b-a) f^{ix}(\xi), \ \xi \in [-1, 1] \quad (29)$$

By mean value theorem "[1]", choosing $(b-a) \le 2$ Hence,

$$E_{RL4GL3}(f) \le \frac{2}{7875 \times 8!}|b-a| \left| f^{ix}(\xi) \right| \le \frac{4M}{7875 \times 8!} \quad (30)$$

Where

$$M = \max_{-1 \le x \le 1} \left| f^{ix}(x) \right| \quad (31)$$

## IV. NUMERICAL VERIFIVATION

For the numerical verification of hybrid quadrature rule $R_{RL4GL3}(f)$, the following integrals are considered and all the mathematical calculations are performed by using MATLAB format long.

$I.\ I_1 = \int_{-1}^{1} e^x \, dx = 2.350402387287603$

| Quadrature rule | Approximation value $I_1$ | Error bound |
|---|---|---|
| $R_{L4GL3}(f)$ | 2.350402491039780 | $\left\|E_{L4GL3}(f)\right\|$ =0.000000103752176 |
| $R_{RExt}(f)$ | 2.350402493664687 | $\left\|E_{RExt}(f)\right\|$ =0.000000106377083 |
| $R_{RL4GL3}(f)$ | 2.350402387658838 | $\left\|E_{RL4GL3}(f)\right\|$ =0.000000000371234 |

$II.\ I_2 = \int_{0}^{\frac{\pi}{4}} \sin x \, dx = 0.292893218813453$

| Quadrature rule | Approximation value $I_2$ | Error bound |
|---|---|---|
| $R_{L4GL3}(f)$ | 0.292893218821979 | $\left\|E_{L4GL3}(f)\right\|$ =0.000000000008526 |
| $R_{RExt}(f)$ | 0.292893218822191 | $\left\|E_{RExt}(f)\right\|$ =0.000000000008783 |
| $R_{RL4GL3}(f)$ | 0.292893218813630 | $\left\|E_{RL4GL3}(f)\right\|$ =0.000000000000177 |

$III.\ I_3 = \int_{0}^{\frac{\pi}{2}} \cos x \, dx = 1.000000000000000$

| Quadrature rule | Approximation value $I_3$ | Error bound |
|---|---|---|
| $R_{L4GL3}(f)$ | 1.000000007959276 | $\left\|E_{L4GL3}(f)\right\|$ =0.000000007959276 |
| $R_{RExt}(f)$ | 1.000000008132813 | $\left\|E_{RExt}(f)\right\|$ =0.000000008132813 |
| $R_{RL4GL3}(f)$ | 1.000000001124582 | $\left\|E_{RL4GL3}(f)\right\|$ =0.000000001124582 |

$IV.\ I_4 = \int_{0}^{\frac{\pi}{4}} \cos^2(x) \, dx = 0.642699081698724$

| Quadrature rule | Approximation value $I_4$ | Error bound |
|---|---|---|
| $R_{L4GL3}(f)$ | 0.64269908368854 | $\left\|E_{L4GL3}(f)\right\|$ =0.000000001989819 |
| $R_{RExt}(f)$ | 0.642699083731927 | $\left\|E_{RExt}(f)\right\|$ =0.000000002033203 |
| $R_{RL4GL3}(f)$ | 0.642699081979882 | $\left\|E_{RL4GL3}(f)\right\|$ =0.000000000281580 |



ISSN: 2278 – 7798 

## V. CONCLUSION

The hybrid quadrature rule $R_{RL4GL3}(f)$ of precision nine is used for evaluating real definite integrals. An asymptotic error estimate and an error bound of the hybrid quadrature rules are given. Finally we have taken some numerical examples to show superiority of the hybrid quadrature rule with respect to the constituent rules. The result due to hybrid quadrature rule $R_{RL4GL3}(f)$ is excellent convergent to the exact value. From the above tables, we observe that,

$$\left|E_{RExt}(f)\right| \leq \left|E_{L4GL3}(f)\right| \leq \left|E_{RL4GL3}(f)\right|$$

## VI. REFERENCE


[1] S.conte and C.de.Boor, “Elementary Numerical Analysis,” Tata Mc-Graw Hill, 1980.

[2] Kendal E.Atkinson, " An Introduction to Numerical Analysis", John Wiley, 2nd edition , 2001.

[3] R.N.Das and G.Pradhan," A mixed quadrature rule for approximate evaluation of real definite integrals", Int.J.Math.edu.sci, vol. 27, pp. 279-283, 1996.

[4] S.R.Jena and R.B.Dash, " Mixed quadrature of real definite integrals over triangles ", Paci-Asian.J.Math , vol. 3, pp. 119-124, 2009.

[5] R.B.Dash and S.R.Jena, " A Mixed quadrature of Modified Birkhoff-Young using Richardson extrapolation and Gauss-Legendre-4 point transformed rule", Int.J.Appli.Math.Appli, vol. 2, pp. 111-117, 2008.

[6] B.P.Acharya and R.N.Dash, " Compound Birkhoff-Young rule for numerical integration of analytic functions", Int.J.Math.Edu.Sci.Technol, vol. 14, pp. 1-10, 1983.

[7] F.G.Lether, " On Birkhoff-Young qadrature of analytic functions", J.Comut.Applied.Math, vol. 2, pp. 81, 1976.

[8] Erwin Kreyszig, " Advanced Engineering Mathematics", Wiley india, 9th edition, 2006.




ISSN: 2278 – 7798 