# Some Hamiltonian Properties and Wiener Index of Graphs

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*Abstract*— The Wiener index of a connected graph is defined as the sum of distances between all pairs of vertices in the graph. L.Yang presented a sufficient condition in terms of the Wiener index for a graph to be traceable. Here we present result based on the Wiener index for a graph to be Hamiltonian or Hamilton-connected in this paper.

**Keywords:** Hamiltonian, Hamiltonian-connected, Wiener Index.
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## I. Introduction

We consider only undirected finite graphs without multiple edges or loops. For a graph G = (V; E), we use n and e to denote its order | V | and size |E| respectively. For both vertices u and v in a graph G, Let $d_G(u, v)$ be denote the distance between them. If a cycle C in a graph G contains all the vertices of G then C is called a Hamiltonian cycle of G. If a graph G has a Hamiltonian cycle then graph G is called Hamiltonian graph. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for every pair of vertices in G there is a Hamiltonian path between them. If G and H are both vertex-disjoint graphs, we use G $\vee$ H to denote the join of G and H. We use C (n, r) to denote the number of r - combinations of a set with n elements.

For a connected graph G, its Wiener index [8], denoted by W (G), it is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$$

If we use $\widehat{D}_G(v)$ to denote $\sum_{u\in V(G)} d_G(u,v)$,

then $W(G)=\frac{1}{2}\sum_{v\in V(G)}\widehat{D}_G(v)$. It can be easily verified that $\widehat{D}_G(v)\ge d(v)+2(n-1-d(v))$.

For a nontrivial connected graph G, its Harary index [5, 7] is defined as $\sum_{\{u,v\}\subseteq V(G)}\frac{1}{d_G(u,v)}$

In [4], Hua and Wang presented a sufficient condition for a graph to be traceable by using Harary index. Li [6] presented sufficient conditions in terms of the Harary index for a graph to be Hamiltonian or Hamilton-connected using some proof ideas in [4]

In [9], Yang presented the following sufficient condition for a graph to be traceable by using Wiener index.

**Theorem 1.1.** [9]. Let G be a connected graph of order n 4. If $W(G)\le\frac{(n+5)(n-2)}{2}$ then G is traceable, unless $G=K_1\vee(K_{n-3}\bigcup 2K_1)$ or $K_2\vee(3K_1\bigcup K_2)$ or $K_4\vee 6K_1$.

In this paper, we combine the ideas in [9] and [6] to present the following sufficient conditions in terms of the Wiener index for a graph to be Hamiltonian or Hamilton-connected.

**Theorem 1.2.** Let G be a connected graph of order n $\ge 4$. If $W(G)\le\frac{n^2+n-6}{2}$ then G is Hamiltonian Connected unless $G=K_2\vee(K_1\bigcup K_{n-3})$ or $K_3\vee(3K_1)$

**Theorem 1.3**. Let G = (X, Y; E), where X = {$x_1, x_2, \ldots, x_n$}, Y = {$y_1, y_2, \ldots, y_n$} and n$\ge$ 2 be a connected bipartite graph. If W (G) $\le 3n^2 - 2n + 2$; then G is Hamiltonian, unless G = $P_4$, a path having four vertices and three edges.

**Theorem 1.4.** Let G be a 2-connected graph of order n $\ge$ 12. If $W(G)\le\frac{n^2+3n-13}{2}$ then G is Hamiltonian, unless $G = K_2\vee((2K_1)\cup K_{n-4})$.

**Theorem1.5.** Let G be a k-connected graph of order n. If $W(G)\le\frac{n(n-1)+(K+1)(n-k-1)-1}{2}$ then G is Hamiltonian.

## II Preliminary Results

**Corollary 2.1.** Let G be a graph of order n $\ge 3$ with degree sequence $d_1\le d_2\le\ldots\le d_n$. If

$d_k\le k<\frac{n}{2}\Rightarrow d_{n-k}\ge n-k,$ then G is Hamiltonian.

**Corollary 2.2.** Let G be a graph of order $n\ge 3$ with degree sequence $d_1\le d_2\le\ldots\le d_n$. If



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$2 \le k \le \frac{n}{2}$, $d_{k-1} \le k \Rightarrow d_{n-k} \ge n-k+1$, then G is Hamilton-connected.

**Corollary 2.3.** Let G = (X, Y ; E) be a bipartite graph such that

$X = \{x_1, x_2, ..., x_n\}$ , $Y=\{y_1, y_2, ..., y_n\}$ $n \ge 2$,

and $d_G(x_1) \le d_G(x_2) \le ... \le d_G(x_n)$,

$d_G(y_1) \le d_G(y_2) \le ... \le d_G(y_n)$.

If $d_G(x_k) \le k < n$

$\Rightarrow d_G(y_{n-k}) \ge n-k+1$,

then G is Hamiltonian.

**Corollary 2.4.** [3] Let G be a 3-connected graph of order $n \ge 18$. If $e(G) \ge C(n-3, 2) + 9$ then G is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.

**Corollary 2.5.** [3] Let G be a k-connected graph of order n. If $e(G) \ge C(n,2) - (k+1)(n-k-1)/2 + 1$ then G is Hamiltonian.

**Note that** Corollary 2.1 is Corollary 3 on Page 208 in [1], Corollary 2:2 is Theorem 12 on Page 218 in [1], Corollary 2.3 is Corollary 5 on Page 210 in [1], and Corollary's 2:4 and 2:5 can be found in [3].

## III Main Results

**Proof of Theorem 1.2.** Let G be a graph which satisfies the conditions in Theorem 1.1. Assume that G is not Hamilton-connected. Then, from corollary 2:2, there exists an integer k such that $d_{k-1} \le k$ *and* $d_{n-k} \le n - k$.

Therefore,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} \widehat{D_G}(v)$$

$$\ge \frac{1}{2} \sum_{v \in V(G)} (d_G(v) + 2(n-1-d_G(v)))$$

$$= \frac{1}{2} \sum_{v \in V(G)} (2(n-1-d_G(v))$$

$$= n(n-1) - \sum_{v \in V(G)} d_G(v)$$

$$\ge n(n-1) - \frac{1}{2}(k(k-1) + (n-2k+1)(n-k) + k(n-1))$$

$$= \frac{n^2+n-6}{2} + \frac{(k-2)(k-3)}{2} + (k-2)(n-2k).$$

such that $W(G) = \frac{n^2+n-6}{2}$

where $k = 2$ or ($k=3$ and $n = 2k$),

$d_1 = \cdots = d_{k-1} = k$, $d_k = \cdots = d_{n-k} = n-k-1$

and $d_{n-k+1} = \cdots = d_n = n-1$.

If $k=2$, then $d_1 = 2$, $d_2 = d_3 \cdots = d_{n-2} = n-2$

and $d_{n-1} = n-1$.

Thus $G = K_2 \vee (K_1 \cup K_{n-3})$,

which is not Hamiltonian.

If k = 3 and n = 2k, then we have that n = 6. Therefore $d_1$ = 3, $d_2$ = 3, $d_3$ = 3, $d_4$ = 5, $d_5$ = 5 and $d_6$ = 5. Hence G = $K_3$ _ ($3K_1$), which is not Hamilton-connected.

This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** Let G be a graph satisfying the conditions in Theorem 1.2. Suppose that G is not Hamiltonian. Then, from corollary 2.3, there exists an integer k < n such that $d_G(x_k) \le k$ and $d_G(y_{n-k}) \le n-k$. Next we find an upper bound for $\widehat{D}_G(x_1)$. Let $N_G(x_1) = \{z_1, z_2, ..., z_s\}$ be the neighbours of $x_1$, where $s = D_G(x_1)$.

Then $d_G(x_1, z_i) = 1$ for each $z_i \in N_G(x_1)$, $d_G(x_1, x_i) \ge 2$ for each $x_i$ with $2 \le i \le n$, and $d_G(x_1, y_i) \ge 3$ for each $y_i$ - $N_G(x_1)$. Thus

$$\widehat{D}_G(x_1) \ge d_G(x_1) + 2(n-1) + 3(n - d_G(x_1))$$
$$= 5n - 2 - 2d_G(x_1).$$

Similarly,

we have that for each $2 \le i \le n$ and each $1 \le j \le n$,

$$\widehat{D}_G(x_i) \ge d_G(x_i) + 2(n-1) + 3(n - d_G(x_i))$$
$$= 5n - 2 - 2d_G(x_i);$$

$$\widehat{D}_G(y_j) \ge d_G(y_j) + 2(n-1) + 3(n - d_G(y_j))$$
$$= 5n - 2 - 2d_G(y_j)$$

Therefore,



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$$W(G) = \frac{1}{2}\sum_{v\in V(G)} \widehat{D_G}(v)$$
$$\geq \frac{1}{2}\left(10n^2 - 4n - 2\sum_{i=1}^{n}(d_G(x_i)+d_G(y_i))\right)$$
$$\geq \frac{1}{2}\left(10n^2 - 4n - 2(\,(k+(n-k))^2 - 2k(n-k)+n^2)\right)$$
$$= \frac{1}{2}\left(10n^2 - 4n - 2((\,k^2 + (n-k)n + (n-k)^2 + kn)\right)$$
$$= \frac{1}{2}\left(10n^2 - 4n - 2(\,(k^2 - 2k(n-k))\right)$$
$$= 3n^2 - 2n + 2k(n-k) + n^2))$$
$$\geq 3n^2 - 2n + 2*1*1$$
$$= 3n^2 - 2n + 2.$$

From $W(G) \leq 3n^2 - 2n + 2,\ 1 \leq k < n,$ we have that

$k = 1$, $n - k = 1$, $d_G(x_1) = 1$, $d_G(x_2) = 2$,

$d_G(y_1) = 1$ and $d_G(y_2) = 2$. Thus $G = P_4$, this is not Hamiltonian.

This completes the proof of Theorem 1.3.

**Proof of Theorem 1.4.** Let G be a graph satisfying the conditions in Theorem 1.3. Note that if G

$K_2 \vee ((2K_1)\bigcup K_{n-4})$, then $W(G) = \dfrac{n^2+3n-14}{2}$.

Assume that G is not Hamiltonian and G is not

$K_2 \vee ((2K_1)\bigcup K_{n-4})$. Then, from corollary 2:4,

we have that $e(G) \leq C(n-2,2)+3.$ So we have,

$$W(G) = \frac{1}{2}\sum_{v\in V(G)} \widehat{D_G}(V)$$
$$\geq \frac{1}{2}\sum_{v\in V(G)} (d_G(V) + 2(n-1-d_G(V)))$$
$$= \frac{1}{2}\sum_{v\in V(G)} (2(n-1-d_G(V))$$
$$= n(n-1) - \frac{1}{2}\sum_{v\in V(G)} d_G(V)$$
$$= n(n-1) - e(G)$$
$$\geq n(n-1) - C(n-2,2) - 3$$
$$= \frac{n^2+3n-12}{2},$$

This is the contradiction to the assumption

This completes the proof of Theorem 1.4.

**Proof of Theorem 1.5.** Let G be a graph satisfying the conditions in Theorem 1:4. Suppose that G is not Hamiltonian. Then, from corollary 2.5, we have that

$$e(G) \leq C(n,\ 2) - (k+1)(n-k-1)/2.$$

Therefore we consider,

$$W(G) = \frac{1}{2}\sum_{v\in V(G)} \widehat{D_G}(v)$$
$$\geq \frac{1}{2}\sum_{v\in V(G)} (d_G(v) + 2(n-1-d_G(v)))$$
$$= \frac{1}{2}\sum_{v\in V(G)} (2(n-1) - d_G(v))$$
$$= n(n-1) - \frac{1}{2}\sum_{v\in V(G)} d_G(v)$$
$$= n(n-1) - e(G)$$
$$\geq n(n-1) - C(n,2) + (k+1)(n-k-1)/2$$
$$= \frac{n(n-1)+(k+1)(n-k-1)}{2},$$

This is the contradiction to the assumption

This completes the proof of Theorem 1.5.

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