Fuzzy Multi-objective Linear Programming Problem Using Fuzzy Programming Model

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Abstract

This paper proposes the method to solve the solution of fuzzy multi-objective linear programming, problem is reduced to crisp using ranking function and then the crisp problem is solved by fuzzy programming technique.

Keywords: Fuzzy Multi-objective Linear programming problem, fuzzy sets, trapezoidal fuzzy numbers, crisp problem, Ranking function.

1. Introduction

Linear programming is one of the most important operational research techniques. It has been applied to solve many real world problems, but it fails to deal with imprecise data, so many researchers succeeded in capturing vague and imprecise information by fuzzy linear programming problem (FLPP) [2,10]. The concept of a fuzzy decision making was first proposed by Bellman and Zadeh [1]. Recently, much attention has been focused on FLPP [9].

An application of fuzzy optimization techniques to linear programming problems with multiple objectives [9] has been presented by Zimmermann.. presented a fuzzy approach to multi-objective linear programming problems. The most common approach to solve fuzzy linear programming problem is to change them to corresponding deterministic linear program. Some methods based on comparison of fuzzy numbers have been suggested by H.R.Maleki [7], A.Ebrahimnejad, S.H.Nasseri [5], F.Roubens [6], L.Campos and A.Munoz[3]. Zimmermann [9] has introduced fuzzy programming approach to solve crisp multi-objective linear programming problem. Recently H.M.Nehi et.al.[8] used ranking function suggested by M.Delgaodo et.al [4] to solve fuzzy MOLPP.

In this paper, we introduced a method in which fuzzy multi-objective linear programming problem is reduced to crisp MOLPP using ranking function suggested by F.Roubens [6] and the resulting one is solved by partial modification of fuzzy programming technique of Zimmermann [10]. The coefficients of all objective functions as well as the constraints are fuzzy in nature. A numerical example is given to illustrate the procedure.

2. Multi-objective Linear Programming

Multi-objective Linear Programming (MOLP) Problems is an interest area of research since most real-life problems have a set of conflict objectives. A mathematical model of the MOLP problem can be written as follows:

$$\text{Max } f(x) = (c_1(x), c_2(x), \ldots\ldots\ldots, c_k(x))$$

$$\text{such that } x \in X = \{ x \in R^n / g_j(x) \le 0 , j = 1,2,\ldots m \} \qquad (2.1)$$

where $(c_1(x), c_2(x), \ldots\ldots\ldots, c_k(x))$ are k distinct linear objective functions of the decision variables and X is the feasible set of constrained decision.

Definition 2.1

X* is said to be a complete optimal solution for (1) if there exist $x^* \in X$ such that $c_i(x^*) \le c_i(x)$, $i = 1, 2 \ldots k$ for all $x \in X$.

ISSN: 2278 – 7798

3. Ranking Function For Fuzzy Numbers

Definition: 3.1

Let A be a fuzzy number whose membership function can generally be defined as

$$\mu_A(x) = \begin{cases} \mu_A^L(x) & a^1 \le x \le a^2 \\ 1 & a^2 \le x \le a^3 \\ \mu_A^R(x) & a^3 \le x \le a^4 \\ 0 & \text{otherwise} \end{cases}$$

Where $\mu_A^L(x) : [a^1, a^2] \rightarrow [0, 1]$ and $\mu_A^R(x) : [a^3, a^4]$ are strictly monotonic and continuous mappings. Then, it is considered as left right fuzzy number. If the membership function $\mu_A(x)$ is piecewise linear, then it is referred to as a trapezoidal fuzzy number and is usually denoted by $A = (a^1, a^2, a^3, a^4)$. If $a^2 = a^3$ the trapezoidal fuzzy number is turned into a triangular fuzzy number $A = (a^1, a^3, a^4)$.

A fuzzy number A = (a, b, c) is said to be a triangular fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 1 & x = b \\ \frac{x-b}{b-c} & b \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

Assume that $R : F(R) \rightarrow R$ be linear ordered function that maps each fuzzy number in to the real number, in which F(R) denotes the whole fuzzy numbers. Accordingly, for any two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ we have

$$\tilde{a} \underset{R}{\ge} \tilde{b} \quad \text{iff} \quad R(\tilde{a}) \ge R(\tilde{b})$$

$$\tilde{a} \underset{R}{\succ} \tilde{b} \quad \text{iff} \quad R(\tilde{a}) > R(\tilde{b})$$

$$\tilde{a} \underset{R}{=} \tilde{b} \quad \text{iff} \quad R(\tilde{a}) = R(\tilde{b})$$

We restrict our attention to linear ranking function R such that

$R(k\tilde{a} + \tilde{b}) = k R(\tilde{a}) + R(\tilde{b})$ for any $\tilde{a}$ and $\tilde{b}$ in F(R) and any $k \in R$.

Roubens' Ranking Function:

The ranking function proposed by F. Roubens is defined by

$$R(\tilde{a}) = 1/2 \int_0^1 (\inf \tilde{a}_\alpha + \sup \tilde{a}_\alpha)\, d\alpha$$

which reduces to

$$R(\tilde{a}) = ½ (a^L + a^U + ½ (\beta - \alpha)). \quad (3.1)$$

For a trapezoidal number

$$\tilde{a} = (a^L - \alpha, a^L, a^U, a^U + \beta)$$

Solving Fuzzy Multi-objective Linear Programming

A fuzzy multi-objective linear programming problem is defined as follows

$$\text{Max} \quad \tilde{z}_r = \Sigma_j \tilde{c}_{rj} x_j \qquad r = 1, 2, \ldots, q$$

$$\text{Such that } \Sigma_j \tilde{a}_{ij} x_j \le \tilde{b}_i \qquad i = 1, 2, \ldots, m \quad (3.2)$$

$$x_j \ge 0$$

where $\tilde{a}_{ij}$ and $\tilde{c}_{rj}$ in the above relation are in the trapezoidal form as

ISSN: 2278 – 7798

$\tilde{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4)$
$\tilde{c}_{rj} = (c_{rj}^1, c_{rj}^2, c_{rj}^3, c_{rj}^4)$

Definition 3.2

$x \in X$ is said to be a feasible solution to the FMOLP problem (3.2) if it satisfies constraints of (3.2).

Definition 3.3

$x^* \in X$ is said to be an optimal solution to the FMOLP problem(3.2) if there does not exist another $x \in X$ such that $\tilde{z}_i(x) \geq \tilde{z}_i(x^*)$ for all I = 1, 2, . . . q.

Now, the FMOLP can be easily transformed to a classic form of a MOLP by considering R as a linear ranking function. By implementing the R on the above model, (3.2) we obtain the classical form of MOLP problem:

$$\text{Max } R(\tilde{z}_r) = \Sigma_j R(\tilde{c}_{rj}) x_j \qquad r = 1, 2, \ldots q$$

Such that $\Sigma_j R(\tilde{a}_{ij}) x_j \leq R(\tilde{b}_j) \qquad i = 1, 2, \ldots m$

$x_j \geq 0$

so we have $\max z_r' = \Sigma_j c'_{rj} x_j \qquad r = 1, 2, \ldots q$

such that

$$\Sigma_j a'_{ij} x_j \leq b'_i \qquad i = 1, 2, \ldots m$$

where a_{ij}', b_i', c_j' are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$, c_j with respect to linear ranking function R respectively.

Lemma 3.4

The optimal solutions of (3.2) and (3.3) are equivalent.

Proof:

let M_1, M_2 be sets of all feasible solutions of (3.2) and (3.3) respectively.

Then $x \in M_1$ iff $\Sigma_j (\tilde{a}_{ij}) x_j \leq (\tilde{b}_j) \qquad i = 1, 2, \ldots m$

By considering R as a linear ranking function, we have

$$\Sigma_j R(\tilde{a}_{ij}) x_j \leq R(\tilde{b}_j) \qquad i = 1, 2, \ldots m$$

$$\Rightarrow \quad \Sigma_j a'_{ij} x_j \leq b'_i$$

Hence $x \in M_2$

Thus $M_1 = M_2$

Let $x^* \in X$ be the complete optimal solution of (3.2)

Then $\tilde{z}_r(x^*) \geq \tilde{z}_r(x)$, for all $x \in X$

where X is the feasible set of solutions.

$$\Rightarrow \quad R(\tilde{z}_r(x^*)) \geq R(\tilde{z}_r(x))$$

$$\Rightarrow \quad (\Sigma R(\tilde{c}_{rj}) x_j^*) \geq (\Sigma R(c_{rj}) x_j), \quad \forall\ j = 1, 2, \ldots q$$

$$\Rightarrow \quad \Sigma c'_{rj} x_j^* \geq \Sigma c_{rj} x_j^*, \quad \forall\ j = 1, 2, \ldots q$$

$$\Rightarrow \quad z_r'(x^*) \geq z_r'(x), \quad \forall\ x$$

$$\mu_{Z_r}(x) = \begin{cases} 0 & \text{if } Z_r \leq L_r \\ \dfrac{Z_r - L_r}{U_r - L_r} & \text{if } L_r < Z_r < U_r \\ 1 & \text{if } Zr \geq Ur \end{cases}$$

Using the above membership functions, we formulate a crisp modal by introducing an augmented variable λ as:

Min : λ

Subject to

$\Sigma c'_{rj} x_j + (U_r - L_r)\lambda \geq U_r, \qquad r = 1, 2, \ldots q$

$\Sigma_j a'_{ij} x_j \leq b'_I, \qquad i = 1, 2, \ldots m$

ISSN: 2278 – 7798

$\lambda \geq 0, \quad x_j \geq 0, \qquad j = 1,2,\ldots n$

4. **Fuzzy Programming Technique**

We have to solve the MOLPP

$$\max z_r' = \Sigma_j c'_{rj} x_j \qquad r = 1, 2, \ldots q \qquad (4.1)$$

$$\text{such that}: \quad \Sigma_j a'_{ij} x_j \leq b'_i \qquad i = 1, 2, \ldots m$$

$$X_j \geq 0$$

In partial modification of Zimmermann's fuzzy programming technique, we formed a technique to solve multi objective linear programming problem. The method is presented briefly in the following steps.

Step -1

Solve the multi objective linear programming problem by considering one objective function at a time and ignoring all others. Repeat the process q times for q different objective functions. Let $x^1, x^2, \ldots x^q$ be the ideal solutions for respective functions.

Step – 2

Using all the above, ideal q solutions in step-1 construct a pay – off matrix of size q by q. The form the pay-off matrix find the lower bound L_r and upper bound (U_r) for the objective function Z_r' as:

$L_r \leq Z_r' \leq U_r \qquad r = 1, 2, \ldots q$

Step-3

Define fuzzy linear membership function $\mu_{Z_r}(x)$ for the objective function Z_r', $r = 1, 2, \ldots q$ as

$$\mu_{Z_r}(x) = \begin{cases} 0 & \text{if } Z_r \leq L_r \\ \dfrac{Z_r - L_r}{U_r - L_r} & \text{if } L_r < Z_r < U_r \\ 1 & \text{if } Zr \geq Ur \end{cases}$$

Using the above membership functions, we formulate a crisp modal by introducing an augmented variable λ as:

Min : λ

Subject to

$$\Sigma c'_{rj} x_j + (U_r - L_r)\lambda \geq U_r, \qquad r = 1, 2, \ldots q \qquad (4.2)$$

$$\Sigma_j a'_{ij} x_j \leq b'_i, \qquad i = 1, 2, \ldots m$$

$$\lambda \geq 0, \quad x_j \geq 0, \qquad j = 1, 2, \ldots n$$

Step - 4

Here the crisp modal is a linear programming problem with linear constraints. This is solved using simplex method. Thus, we get compromise solution.

5. Numerical Example

Max: $\tilde{z}_1(x) = \tilde{3}x_1 + \tilde{2}x_2$

Max: $\tilde{z}_2(x) = \tilde{2}x_1 + \tilde{3}x_2$

Such that

$\tilde{1}x_1 + \tilde{1}x_2 \leq \tilde{4}$

$\tilde{1}x_1 - \tilde{1}x_2 \leq \tilde{2}$

$x_1, x_2 \geq 0$

$\tilde{3} = (2.2, 2.3, 3.3, 3.8)$

$\tilde{2} = (1.9, 2.1, 2.2, 2.6)$

$\tilde{2} = (1.2, 1.9, 2.3, 2.8)$

$\tilde{3} = (2.2, 2.8, 3.9, 3.5)$

ISSN: 2278 – 7798

$\tilde{1}$= (0.9, 0.6, 0.7, 0.5)

$\tilde{1}$= (0.8, 0.6, 0.9, 1.7)

$\tilde{1}$ = (1.8, 0.7, 0.9, 1.5)

$\tilde{1}$ = (1.4, 0.8, 0.9, 0.6)

Using Ranking function suggested by F.Roubens [6] the problem reduces to

$z_1'(x) = R(\tilde{3})\, x_1 + R(\tilde{2})\, x_2$

$z_2'(x) = R(\tilde{2})\, x_1 + R(\tilde{3})\, x_2$

such that

$R(\tilde{1})\, x_1 + R(\widetilde{1})x_2 \leq R(\tilde{4})$

$R(\tilde{1})\, x_1 - R(\tilde{1})\, x_2 \leq R(\tilde{2})$

$x_1, x_2 \geq 0\ \tilde{z}$

$z_1'(x) = 3.3\, x_1 + 2.2\, x_2$ (1)

$z_2'(x) = 1.9\, x_1 + 3.4\, x_2$ (2)

such that

$1.1\, x_1 + 0.9\, x_2 \leq 3.8$ (3)

$1.2\, x_1 - 1.1\, x_2 \leq 2.1$

$x_1, x_2 \geq 0.$

Solving (1) with (3) by Simplex method

$x_1 = 2.25$ $x_2 = 1.47$

Solving (2) with (3)by Simplex method

$x_1 = 0$ $x_2 = 4.22$

The lower bound (L.B) and upper bound (U.B) of objective functions z_1' and z_2' have been completed as follows: Function

	L.B	U.B
z_1'	9.284	10.659
z_2'	9.273	14.348

Min : λ

Such that

$3.3x_1 + 2.2x_2 + 1.375 \geq 10.659$ (4.3)

$1.9x_1 + 3.4x_2 + 5.075 \geq 14.348$

$1.1x_1 + 0.9x_2 \leq 3.8$

$1.2x_1 - 1.1x_2 \leq 2.1$

$x_1 , x_2 \geq 0$

Solving $x_1^* = 2.78$

$x_2^* = 1.978$

Now the optimal values of the objective functions of FMOLPP (4.3) becomes,

$\tilde{z}_1^* = 3.3x_1^* + 2.2x_2^*$

$\tilde{z}_2^* = 1.9x_1^* + 3.4x_2^*$

$\tilde{z}_1^* = (2.2, 2.3, 3.3, 3.8)\, x_1^* + (1.9, 2.1, 2.2, 2.6)x_2^*$

ISSN: 2278 – 7798

= (9.9742, 10.5478, 13.5256, 15.7068)

$\tilde{z}_2^* = (1.2, 1.9, 2.3, 2.8)\, x_1^* + (2.2, 2.8, 3.9, 3.5) x_2^*$

= (7.6876, 10.8204, 14.1082, 14.707)

The membership functions corresponding to the fuzzy objective functions are as follows:

$$\mu_{\tilde{z}_1}(x) = \begin{cases} 0 & x \leq 9.8742 \\ \frac{x - 9.8742}{0.6736} & 9.8742 < x \leq 10.5478 \\ 1 & 10.5478 < x \leq 13.5256 \\ \frac{15.7068 - x}{2.1812} & 13.5256 < x \leq 15.7068 \\ 0 & x > 15.7068 \end{cases}$$

$$\mu_{\tilde{z}_2}(x) = \begin{cases} 0 & x \leq 7.6876 \\ \frac{x - 7.6876}{3.1328} & 7.6876 < x \leq 10.8204 \\ 1 & 10.8204 < x \leq 14.1082 \\ \frac{14.707 - x}{0.5988} & 14.1082 < x \leq 14.707 \\ 0 & x > 14.707 \end{cases}$$

6. Conclusion

In this paper, we discussed the solution of fuzzy multi-objective linear programming problem with the help of objective constrained linear programming problem. The method can be applied to problems when both objective functions and constraints are nonlinear.

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ISSN: 2278 – 7798