Some quadrature methods for approximate evaluation of definite integrals for Complex Variables

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Abstract
In this paper, two quadrature rules Weddle’s transformed rule and Clenshaw-Curtis five point rule of same precision five are mixed up and a quadrature rule of higher precision seven is obtained. An asymptotic error estimate of the rule has been determined and the rule is numerically verified with suitable examples.

Keywords: Quadrature rule, Asymptotic error, Analytic function, Numerical integration, Taylor’s series.

MSC2010: 65D30,65D32

1. Introduction
Keeping the facts in mind [3,4,5,6,9,10,11] we have mixed up Weddle’s transformed rule and Clenshaw-Curtis five point rule each of same precision five and produced a mixed quadrature of precision seven. A new rule of precision seven is used for evaluating an integral of the form,

\[ I(f) = \int f(z)dz. \] (1.1)

Where \( L \) is a directed line segment from the point \((z_0 - h)\) to \((z_0 + h)\) in the complex plane and \( f(z) \) is analytic in certain domain \( \Omega \) containing the line segment \( L \). Lether [7] using the transformation \( z = (z_0 + th) \), where \( t \in [-1, 1] \) transformed the integral (1.1) to the integral

\[ h \int_{-1}^{1} f(z_0 + th)dt. \] (1.2)

2. Construction of mixed quadrature rule:
Weddle’s transformed rule \( R_w(f) \) is

\[ I(f) \approx R_w(f) = \frac{h}{10} \left[ f(z_0 - h) + f(z_0 + h) + 2f(z_0 + h/3) + 5f(z_0 - 2h/3) + 8f(z_0) \right] \] (2.1)

Expanding each term of eqn (2.1) using Taylor series about \( z_0 \)

\[ f(z_0) + \frac{h^2}{3!} f''(z_0) + \frac{h^4}{5!} f^{(iv)}(z_0) + \frac{35h^6}{243 \times 6!} f^{(vi)}(z_0) + \frac{1307h^8}{10935 \times 8!} f^{(vii)}(z_0) + \ldots \]

We can write eqn (1.1) using Taylor series expansion about \( z_0 \)

\[ I(f) = 2h \left[ f(z_0) + \frac{h^2}{3!} f''(z_0) + \frac{h^4}{5!} f^{(iv)}(z_0) + \frac{h^6}{7!} f^{(vi)}(z_0) + \frac{h^8}{9!} f^{(vii)}(z_0) + \ldots \right] \] (2.3)

Cleanshaw-curtis five point rule is

\[ I(f) \approx R_{CC5}(f) = \frac{h}{15} \left[ f(z_0 + h) + f(z_0 - h) + 8f\left( z_0 + \frac{h}{\sqrt{2}} \right) + 8f\left( z_0 - \frac{h}{\sqrt{2}} \right) + 12f(z_0) \right] \] (2.4)
3. Error Analysis

Now the error associated with Weddle’s rule $R_w(f)$ is

$$E_w(f) = I(f) - R_w(f) = \frac{-4h^7}{1701 \times 6!} f^w(z_5) - \frac{184h^9}{10935 \times 8!} f^{iii}(z_0)$$  \hspace{1cm} (3.1)

Now the error associated with Clenshaw-Curtis five point rule $R_{CCS}(f)$ is

$$E_{CCS}(f) = \frac{1}{315 \times 5!} h^7 f^{w}(z_0) + \frac{1}{360 \times 7!} h^9 f^{iii}(z_0) + \ldots$$  \hspace{1cm} (3.2)

Using Taylor series expansion of eqn (2.1) and eqn (2.4) we get,

$$I(f) = R_w(f) + E_w(f)$$  \hspace{1cm} (3.3)

$$I(f) = R_{CCS}(f) + E_{CCS}(f)$$  \hspace{1cm} (3.4)

Where $f$ is infinitely differentiable since it is assumed to be analytic in certain domain $\Omega$ containing the line segment $L$.

Now multiplying $\left( \frac{1}{81} \right)$ in eqn (3.4) and $\left( \frac{1}{10} \right)$ in eqn (3.3) and adding,

$$I(f) = R_{WCCS}(f) + E_{WCCS}(f)$$  \hspace{1cm} (3.5)

$$R_{WCCS}(f) = \frac{1}{91} \left[ 10 R_{CCS}(f) + 81 R_w(f) \right]$$  \hspace{1cm} (3.6)

$$E_{WCCS}(f) = \frac{1}{91} \left[ 10 E_{CCS}(f) + 81 E_w(f) \right]$$  \hspace{1cm} (3.7)

**Calculation of $E_{WCCS}(f)$**

Substituting eqn (3.1) and eqn (3.2) in eqn (3.7)

$$E_{WCCS}(f) = -\frac{77}{49140 \times 7!} h^9 f^{iii}(z_0)$$  \hspace{1cm} (3.8)

From eqn (3.8), the degree of precision of $R_{WCCS}(f)$ is seven where as degree of precision of $R_w(f)$ and $R_{CCS}(f)$ are five.

4. Numerical Verification

Let us approximate the value of the following integrals $I_1$, $I_2$, $I_3$ using $R_w(f)$, $R_{CCS}(f)$, $R_{WCCS}(f)$ quadrature rule

Where

$$I_1 = \int_{-i}^{i} e^z \, dz, \quad I_2 = \int_{-i}^{i} \cos z \, dz, \quad I_3 = \int_{-i}^{i} \cosh z \, dz$$

**Table-1 (Comparison between approximate and exact results)**

<table>
<thead>
<tr>
<th>Exact value of the integrals</th>
<th>Weddle’s Transformed rule $R_w(f)$</th>
<th>Clenshaw-Curtis five point rule $R_{CCS}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$ = 1.6829419</td>
<td>1.6829391</td>
<td>1.6829678776630</td>
</tr>
<tr>
<td>69615793i</td>
<td>11213524i</td>
<td>91i</td>
</tr>
<tr>
<td>$I_2$ = 2.3504023</td>
<td>2.3504060</td>
<td>2.3503753769314</td>
</tr>
<tr>
<td>87287603i</td>
<td>80572671i</td>
<td>79i</td>
</tr>
<tr>
<td>$I_3$ = 0.6543893</td>
<td>0.6543893</td>
<td>0.6543894056608</td>
</tr>
<tr>
<td>93592304i</td>
<td>92120058i</td>
<td>18i</td>
</tr>
</tbody>
</table>

**Table-2 (Comparison between approximate and exact results)**

| Mixed quadrature rule $R_{WCCS}(f)$ | $|E_w(f)|$ | $|E_{CCS}(f)|$ | $|E_{WCCS}(f)|$ |
|-------------------------------------|----------|--------------|----------------|
| 1.682942272                        | 0.00002  | 0.00002      | 0.00000        |
| 361828i                             | 285      | 590          | 0.300          |
| 2.350407206                        | 0.00000  | 0.00002      | 0.00000        |
| 546167i                             | 369      | 701          | 0.031          |
| 0.654389393                        | 0.00000  | 0.00000      | 0.00000        |
| 608054i                             | 001      | 001          | 00000          |

From the Table-1 and Table-2 we compare the results as follows,

$$|E_{WCCS}(f)| \leq |E_w(f)| \leq |E_{CCS}(f)|$$

5. Conclusion

From the Table-1 and Table-2 it is evident that the mixed quadrature rule $R_{WCCS}(f)$ of degree of precision seven is giving us better result than the constituent rules $R_w(f)$ and $R_{CCS}(f)$ each of degree of precision five. Our quadrature rule is more efficient and numerically better convergent to exact result.
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References


