On Spectral Theory Of K-n- Arithmetic Mean Idempotent Matrices On Posets

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***Abstract*— We Consider idempotent and n-idempotent matrices and corresponding examples are introduced. Let C^n-be the unitary space of order n, C_{nxn} – the set of all complex nxn – matrices, K – the fixed product disjoint transposition is S_n and S_n – the set of all permutation matrices on {1,2,3,..........,n}. We define K – idempotent and k-n idempotent matrices on spectral theory as an abstract generalization of k-n idempotent matrices on C_{nxn} and we define arithmetic mean idempotent matrices. Some of the most important properties of K- eigen values are presented in terms of K-idempotent matrices.**

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***Index Terms*— A.M.S. Classification: 15A57, 11B25, 11C2. Matrices, idempotent (or) di-potent, K – idempotent and k-n – idempotent matrices, K- eigen values and spectral theory properties and arithmetic mean idempotent matrices and K-arithmetic mean idempotent matrices on posets.**

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1. INTRODUCTION

A K-idempotent matrix is defined and some of its basic characterization are derived, [2]. It is shown that if A is a K-idempotent matrix then it is quadripotent (ie, A^4=A).

Necessary and sufficient condition for the sum of two K-idempotent matrices to be K-idempotent, is determined and then it is generalized for the sum of 'n' K – idempotent matrices[1]. A condition for the product of two K – idempotent matrices to be K- idempotent is also determined and then it is generalized for the product of 'n' K – idempotent matrices.

Relation between power hermit an matrices ($A^*=A^n$) and K – idempotent matrices are investigated[3]. It is proved that a K – idempotent matrices A reduces to an idempotent matrix when it commutes with the associated permutation matrix K (ie, AK = KA) and also defined K – eigen values and K – eigen vectors of a complex matrices as a specialization of generalized eigen values problem Ax = λBx.

2. n – idempotent matrices.

Definition:

An idempotent matrix is a matrix in which when multiplied by itself, yields itself.

i.e, The matrix M is idempotent if and only if MM=M. For this Product MM to be defined, M must necessarily be a Square matrix.

2.1 Example:

Example of 2x2 idempotent matrix is $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Example of 3x3 idempotent matrix is $\begin{pmatrix}2 & -2 & -4\\-1 & 3 & 4\\1 & -2 & -3\end{pmatrix}$

2.2 Definition:

A Square matrix A is said to be n – idempotent matrix if A^n = A for any positive integer n. In general, A^{P+1}= A for any positive integer P.

Example :

$$A = \begin{pmatrix}2 & -2 & -4\\-1 & 3 & 4\\1 & -2 & -3\end{pmatrix}$$

here A^2 =A, A^3 =A,......, A^h = A.

3. On n-k idempotent matrices:

Definition

For a fixed product of disjoint transpositions K∈S_n, a matrix A= <aij> in C^{nxn} is said to be K –idempotent if $\sum_{t=1}^{n} a_{k(i)t}a_{tk(j)} = a_{ij}$ is equivalent to KA^2K =A, Where K is the associated permutation matrix of 'K'.

Example:

$$A = \begin{pmatrix}½ & 0 & 0 & ½\\0 & -2 & -\sqrt{3} & 0\\0 & \sqrt{3} & 1 & 0\\½ & 0 & 0 & ½\end{pmatrix}$$

Then,

$$\text{If } A^2 = \begin{pmatrix}½ & 0 & 0 & ½\\0 & -1 & \sqrt{3} & 0\\0 & -\sqrt{3} & -2 & 0\\½ & 0 & 0 & ½\end{pmatrix}$$

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Here A is a K – idempotent matrix with $K = \langle 1,4\rangle \langle 2,3\rangle$

The associated permutation matrix is a matrix with ones on its south west – north east diagonal and zeros everywhere else.

i.e, $$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

It can be easily verified that $KA^2K = A$

3.1 Remark:

$KA^2K = A$ implies that $KAK = A^2$. The following relations can also be obtained which would be useful in computational aspects.

$$KA = A^2K \text{ (or)}$$
$$KA^2 = AK$$
$$KA^3 = A^3K \text{ (or)}$$
$$KA^3K = A^3$$
$$A^3 = (KA)^2 \text{ (or) } (AK)^2$$

3.2 Theorem:

Let A and B two K – idempotent matrices. Then A + B is K – idempotent if and only if AB = - BA

Proof:

$$\begin{aligned} A+B &= KA^2K + KB^2K \\ &= K(A^2+B^2)K \\ &= K(A+B)^2K \quad \text{iff } AB = -BA \end{aligned}$$

3.3 Theorem:

Let A and B be K-idempotent matrices. If AB=BA then AB is the also be K-idempotent matrix.

Proof:

$$\begin{aligned} AB &= (KA^2K)(KB^2K) \\ &= KA^2B^2K \\ &= KAABBK \\ &= K(AB)^2K. \quad \text{(by } AB = BA\text{)} \end{aligned}$$

Hence the matrix AB is K - idempotent

3.4 Theorem:

Let A be a K-idempotent matrix. Then I-A is K-idempotent iff A is idempotent.

Proof:

$$\begin{aligned} I-A &= K(I-A)^2K \\ &= K(I-2A+A^2)K \\ &= I - 2A^2+A \end{aligned}$$

$\Rightarrow -2A^2+A+A=0$

$\Rightarrow 2A-2A^2=0 \Rightarrow 2(A-A^2)=0.$

$\Rightarrow$ A is idempotent

Conversely, if A is idempotent then A commutes with the permutation matrix K. (cf. lemma 2.2.5)

$$\begin{aligned} K(I-A)^2K &= K(I-2A+A^2)K \\ &= K(I-A)^2K \\ &= K(I-A)K \\ &= I-A \end{aligned}$$

Hence, I-A is K –idempotent.

4. K – idempotent matrix:

Let, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$(KA)^2 = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right]^2$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$= KA$

Similarly for, $(KA)^n = KA$.

It is an n-k idempotent mean matrices

4.1 Eigen Value:

$$|KA - \lambda I| = 0$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow$ **$\lambda = 0,1$**

Eigen values of mean and KA mean matrices are always same.

5. Arithmetic mean idempotent matrices on posets:

In this mean, it satisfies the condition as follows.

$$\frac{A+B}{2} = \left(\frac{A+B}{2}\right)^2$$

5.1 Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\frac{A+B}{2} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right]$$

$$\left(\frac{A+B}{2}\right)^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

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$$= \frac{2}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right]$$

$$\frac{A+B}{2} = \left(\frac{A+B}{2}\right)^2$$

5.2 K – Arithmetic mean idempotent matrices on posets:

In this case, it satisfies the conditions as follows

$$K\left(\frac{A+B}{2}\right)K = K\left(\frac{A+B}{2}\right)^2 K$$

5.3 Example:

If $K = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $A = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$,

$B = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

$$K\left(\frac{A+B}{2}\right)K$$

$$= \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$K\left(\frac{A+B}{2}\right)^2 K$$

$$= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{4}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{2}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

Hence,

$$K\left(\frac{A+B}{2}\right)K = K\left(\frac{A+B}{2}\right)^2 K$$

5.4 Arithmetic mean of n-k – idempotent matrix on posets:

In this case, it satisfies the condition as follows.

$$K\left(\frac{A+B}{2}\right)K = K\left(\frac{A+B}{2}\right)^n K$$

5.5 Example:

If $K = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $A = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $B = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$

$$K\left(\frac{A+B}{2}\right)^3 K = K\left(\frac{A+B}{2}\right)^2 K.\ K\left(\frac{A+B}{2}\right)K$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$= K\left(\frac{A+B}{2}\right)K$$

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$$K\left(\frac{A+B}{2}\right)^n K = K\left(\frac{A+B}{2}\right)K$$

5.6 Theorem:

Let A and B be two Arithmetic mean K –idempotent matrices. The A+B is Arithmetic mean K – idempotent matrices where $A=\frac{A+B}{2}$ and $B=\frac{C+D}{2}$

Proof:

$$A+B = \left(\frac{A+B}{2}\right)+\left(\frac{C+D}{2}\right)$$

$$= K\left(\frac{A+B}{2}\right)^2 K + K\left(\frac{C+D}{2}\right)^2 K$$

$$= K\left[\left(\frac{A+B}{2}\right)^2 + \left(\frac{C+D}{2}\right)^2 K\right]$$

$$= K\left[\left(\frac{A+B}{2}\right) + \left(\frac{C+D}{2}\right)\right]^2 K \quad \text{iff } AB = -BA$$

$= K(A+B)^2 K$ iff AB = - BA

Hence, the matrix A+B is arithmetic mean K- idempotent matrix.

5.7 Theorem:

Let A & B be Arithmetic Mean K – idempotent matrices. If AB=BA then AB is also be Arithmetic mean K – idempotent matrix, where $A=\frac{A+B}{2}$,

$B = \frac{C+D}{2}$.

Proof:

$$AB = \left(\frac{A+B}{2}\right)\cdot\left(\frac{C+D}{2}\right)$$

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$$= K\left(\frac{A+B}{2}\right)^{2} K.\ K\left(\frac{C+D}{2}\right)^{2} K$$

$$= K\left(\frac{A+B}{2}\right)^{2}.\left(\frac{C+D}{2}\right)^{2} K$$

$$= K\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right).\left(\frac{C+D}{2}\right)\left(\frac{C+D}{2}\right) K$$

$$= K\left(\left(\frac{A+B}{2}\right).\left(\frac{C+D}{2}\right)\right)^{2} K$$

(by AB=BA)

$= K(AB)^2K$

$= K\ (AB)^2K$

Hence, the matrix AB is arithmetic mean k-idempotent.

5.8 Theorem:

If A is arithmetic mean K-idempotent matrix then A, A^T, A^{-1} are also arithmetic mean K –idempotent matrices, where $A = \frac{A+B}{2}$.

Proof:

i) $A = KA^2K \Rightarrow \left(\frac{A+B}{2}\right) = K\left(\frac{A+B}{2}\right)^{2} K$

$$\Rightarrow \left(\frac{A+B}{2}\right)^{-1} = \left(K\left(\frac{A+B}{2}\right)^{2} K\right)^{-1}$$

$$= K\left(\left(\frac{A+B}{2}\right)^{2}\right)^{-1} = K\left(\left(\frac{A+B}{2}\right)^{-1}\right)^{2} K$$

Thus, A^{-1} is arithmetic mean K –idempotent matrix

ii) $A = KA^2K \Rightarrow \left(\frac{A+B}{2}\right) = K\left(\frac{A+B}{2}\right) K$

$$\left(\frac{\overline{A+B}}{2}\right) = \left[\overline{K\left(\frac{A+B}{2}\right)^{2} K}\right]$$

$$= K\left(\frac{\bar{A}+\bar{B}}{2}\right)^{2} K$$

Thus, $\bar{A}$ is arithmetic mean K – idempotent matrix

iii) $A = KA^2\ K \Rightarrow \left(\frac{A+B}{2}\right) = K\left(\frac{A+B}{2}\right)^{2} K$

$$\Rightarrow \left(\frac{A+B}{2}\right)^{T} = \left(K\left(\frac{A+B}{2}\right)^{2} K\right)^{T}$$

$$= K\left(\left(\frac{A+B}{2}\right)^{2}\right)^{T} K$$

$$= K\left(\left(\frac{A+B}{2}\right)^{T}\right)^{2} K$$

$$= K\left(\frac{A^T+B^T}{2}\right)^{2} K$$

Thus, A^T is arithmetic mean K – idempotent matrix.

5.9 Eigen values of Arithmetic mean matrices:

$$|(A-\lambda I)| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda\,(1-\lambda) = 0$

$\boldsymbol{\lambda = 0,1}$

$$|(B-\lambda I)| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda\,(1-\lambda) = 0$

$\boldsymbol{\lambda = 0, 1}$

Eigen values of Arithmetic mean idempotent matrix is,

$$\left|\frac{A+B}{2} - \lambda I\right| = 0$$

$$\left|\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$(½-\lambda)^2 - ¼ = 0$$

$$(½-\lambda + ½)\,(1/2 - \lambda - ½) = 0$$

$$\lambda = 0,1$$

Eigen values of Arithmetic mean idempotent matrices are always same.

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REFERENCES

[1] J. K. Baksalary, and O. M. Baksalary, "Idempotency of linear combinations of two idempotent matrices", *Linear Algebra Appl*., pp. 3-7, 321 (2000).

[2] S. Krishnamoorthy, and T. Rajagopalan, "On K- idempotent matrices", *Int. Rev.Pure Appl. Math*., pp. 97-101, 5(1) (2009).

[3] S. Krishnamoorthy, and T. Rajagopalan, " Spectral and K – Spectral theory of K –idempotent matrices", *Int. J. Math. Stat*, pp. 81-85, 7 (10), winter (2010).

[4] David A .Harville , "Matrix Algebra From a Statistician's Perspective".

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