S- Prime Meet Matrices on posets

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Abstract - **We consider S-prime meet matrices as an abstract generalization of S-prime greatest common divisor (GCD) matrices . We also found determinant and inverse and discuss the some of the most important properties of S-prime GCD matrices are presented interms of** Error! Reference source not found. **meet me meet matrices .**

***Index Terms* - Lattice,** Error! Reference source not found. **meet, arithmetical functions andMobiusfunction. Mathematics subject classification: 11AXX,11A0S,11A25 and11A41.Computer classification: MS2007.**

1. INTRODUCTION

Let S = {$x_1, x_2, x_3,\ldots,x_n$ } be a set of n positive integers with $x_1 < x_2 < x_3 < \ldots < x_n$ and let $f : p \to \mathbb{C}$ be a complex valued function on Z_+ (i.e., arithmetical function). Let (x_i ,x_j) denotes the greatest common divisor (GCD) of x_i and x_j and define the nxn matrices $(S)_f = ((S)_f)_{i,j} = f(x_i, x_j)$ .We refer to $(S)_f$ as the GCD matrix on S with respect to f . The set S is said to be factor closed if it contains every positives divisor of each $x_i \in$ S clearly a factor closed set is always GCD – closed further converse does not hold.

In 1876, the concept of classical Smith determinant with entries on Z_+ was introduced by H.J.S .Smith [12] is,

$$\det\ [(xi\ ,xj)]_{n\times n} = \phi(x_1).\phi(x_2).\phi(x_3)..\phi(x_n)$$
$$= \prod_{i=1}^{n} \Phi(x_i)$$

where ϕ is the Euler's totient function . The GCD matrix with respect to f is,

$$(f(x_i,x_i)) = \begin{bmatrix} f(x_1,x_1) & f(x_1,x_2) & ,...,\ f(x_1,x_n) \\ f(x_2,x_1) & f(x_2,x_2) & ,...,\ f(x_2,x_n) \\ . & . & . \\ . & . & . \\ . & . & . \\ f(x_n,x_1) & f(x_n,x_2) & ,...,\ f(x_n,x_n) \end{bmatrix}$$

and $\det\ [f(xi\ ,xj)] = \prod_{k=1}^{n} (f*\mu)(x_k)$

In [1992] , S.Beslin and S.Ligh (7) generalized in this results on GCD matrices by showing that the determinant of the GCD Matrix on a GCD closed set S = {$x_1, x_2, x_3,\ldots,x_n$ } is the product

$$\prod_{k=1}^{n} (\alpha_k) \quad \text{Where } \alpha_k = \sum_{\substack{d/x_k \\ d \nmid x_t \\ x_t < x_k}} \Phi(d)$$

Let S = {$x_1, x_2, x_3,\ldots,x_n$ } be a set of distinct positive integer and a $n \times n$ matrix $(S)_f = (S_{ij})$ clearly $(S_{ij}) = 4(x_i, x_j) + 1$,call it to be S - prime GCD matrix on S . If S is factor closed set then ,det $(S)_j = \prod_{i=1}^{n} g(x_i)$

where $g(n) = \sum_{d/n} (4d+1)(\mu(n/d))$

In this paper describes an abstract generalization of S-prime GCD matrices, namely S-prime meet matrices on posets. Previously results in this direction were obtained in [1, 2, 3, 5, 9, 10, 11]. The purpose of the paper is to express sum of the most important properties of S-prime GCD –matrices on a factor closed sets in the language of S-prime meet matrices, more precisely to set a structure theorem for S-prime meet matrices then derive explicit expression and found further determinant and inverse of S-prime meet matrices.

2.Structure of S-prime Meet Matrices on posets

2.1 Definition

Let $(p, \preceq) = (Z^+, |)$ be a finite poset. We call P be a meet - semi lattice if for any x, y $\in$ p there exist a unique z $\in$ p. such that (i) z$\leq$ x and z $\leq$ y and (ii) If w $\leq$ x and w $\leq$ y for some w$\in$ p .then w $\leq$z. In such a case z is called the meet of x and y is denoted by $x \Lambda y$. For each x$\in$ p, the principal order ideal$\downarrow$ x is defined by $\downarrow$ x ={y$\in$ p/y$\leq$ x} p.246, [8]

2.2 Definition

Let S be a subset of subset of P .we call S be a lower- closed if for every x, y$\in$ p and x$\in$ S and y$\leq$ S .we have y$\in$ S.

2.3 Definition

Let S be a subset of P then S is said to be meet-closed if for every x, y$\in$ S we have $x \wedge y \in$ S.

In this case S itself is a meet lattices. It is clear that a lower –closed subset of a meet semi- lattice is always meet-closed but not conversely. The concept "lower–closed" and "Meet -closed" are generalization of "factor- closed" and "GCD-closed" [6,7] respectively .

In what follows, let P always denotes a finite meet lattice, S a poset that can be embedded in a Meet- semi lattice and $\bar{s}$ the unique minimal meet semi-lattice containing S.

2.4 Definition

Let x and y be two elements the poset P and μ is the mobius function of the poset (S, $\prec$) then

$$\mu(x,y)=\begin{cases} 0 & if \quad x\neq y \\ 1 & if \quad x=y \\ -\sum_{z:z\leq y}\mu(x,z) & otherwise \end{cases}$$

2.5 Definition

Let $(p,\prec,\wedge)$ be a S-prime meet-semi lattice,

let S={$x_1, x_2, x_3,\ldots,x_n$ } be a subset of P such that $x_i \prec x_j \Rightarrow i<j$ and let f be a complex – valued function on P . Then n $\times$ n matrix $(s)_f=((s)_f)_{i,j}=(f_{i,j})$ where $f_{i,j}=4(x_i\wedge x_j)+1$ is called the S-prime Meet Matrix on S with respect to f.

3.Generalized Totient functions

3.1 Definition

let S = {$x_1, x_2, x_3,\ldots,x_n$ } be a subset of P, and let f be a function on P with complex values .Then the function $g_{s,f}$ on S is defined inductively by

$$g_{s,f}(x_j)=f(x_j)-\sum_{x_i\leq x_j}g_{s,f}(x_i)$$

Where $x_i<x_j$ means that $x_i<x_j$, $x_i \neq x_j$ or

$$f(x_j)=\sum_{x_i\leq x_j}g_{s,f}(x_i) \quad (p.2,[2])$$

3.2 Remark

If S is a factor –closed set of positive integer ordered by divisibility and $f(x)=x$ for all x, then $g_{s,f}=\Phi$, Euler's totient function. Thus $g_{s,f}$ in definition 3.1 is a generalization of Euler's totient function

3.3 Theorem

Let S={$x_1, x_2, x_3,\ldots,x_n$ } be S-prime Meet-closed . Without loss of generality we may assume that i<j whenever $x_i<x_j$, then

$$g_{s,f}(x_j)=\sum_{z\leq x_j}\sum_{\substack{w\leq z\\ z\leq x_t\\ t<j}}f(w)\mu(w,z)$$

Where μ is the mobius function of P.

Proof:

By using the definition 3.1

$$f(x_j)=\sum_{x_i\leq x_j}g_{s,f}(x_i)=\sum_{x_i\leq x_j}\sum_{\substack{z\leq x_j\\ z\not\leq x_t\\ t<j}}\sum_{w\leq z}f(w)\mu(w,z)$$

We write,

$$f(x)=\sum_{z\leq x}g(z) \; or\; g(x)=\sum_{z\leq x}f(z)\mu(z,x)$$

for all $x\in P$

It has to be prove that,

$$\sum_{z\leq x_j}g(z)=\sum_{x_i\leq x_j}\sum_{\substack{z\leq x_i\\ z\not\leq x_t\\ t<i}}g(z)$$

Now consider the sum of R.H.S of equation (1)

Let $x_i\leq x_j$ and $z\leq x_i \Rightarrow z\leq x_j$.

Thus every z occurring on the right side of equation (1) occurs on the left side of equation (1).

Conversely, consider the sum on the left side of equation (1).

Suppose that $z\leq x_j$ we have $z\leq x_i$ by minimality of i, we have r = i or $x_r = x_i$, therefore $x_r\leq x_j$ means $x_r\leq x_j$ thus every z occurring on the side of equation (1).

This completes the proof .

3.4 Theorem

If S is lower closed subset of p.

Then $g_{s,f(x_j)}=\sum_{x_i\leq x_j}f(x_i)\mu(x_i,x_j)$

Proof:

Already we know that the result,

$$g_{s,f(x_j)}=\sum_{\substack{z\leq x_j\\ z\not\leq x_t\\ t<j}}\sum_{w\leq z}f(w)\mu(w,z)$$

It reduces we get the proof of theorem Then S is lower closed.

3.5 Example

Let S={$x_1,x_2,\ldots\ldots.x_n$} be a chain with $x_1<x_2<\ldots\ldots<x_n$. Then$g_{s,f}(x_1)=f(x_1)$, $g_{s,f}(x_2)=f(x_2)-f(x_1)$

In general $g_{s,f}(x_j)=f(x_j)-f(x_{j-1})$ where, j=2, 3, 4,…, n.

3.6 Example

Let S={$x_1 ,x_2 ,\ldots..x_n$ } be an incomparable set and let S={$x_0, x_1 ,x_2 ,\ldots..x_n$ }. Then, $g_{s,f}(x_0)=f(x_0)$,

$g_{s,f}(x_1)=f(x_1)-f(x_0)$, and

$g_{s,f}(x_2)=f(x_2)-f(x_0)$.

In general $g_{s,f}(x_j)=f(x_j)-f(x_0)$

for j=1, 2, 3, …,n

3.7 Theorem (STRUCTURE THEOREM)

Let S= {$x_1, x_2\ldots x_n$ } and T= {$y_1, y_2\ldots y_m$ } be any two subsets of P. Define the incidence matrix whose i, j-entry is 1 if $y_j\leq x_i$ and zero otherwise namely that is,

E(S, T) = $(e_{i,j})_{n\times m}$

where $(e_{i,j})=\begin{cases}1 & ,if \quad y_j\leq x_i\\ 0 & ,if \quad otherwise\end{cases}$

Example:

1.We consider S={ 5,9,13},T={ 9,17,21} are the S-prime number subsets.

ISSN: 2278 – 7798

Then the incidence matrix of (S, T) is E(S, T)

$$=(e_{i,j})=\begin{bmatrix}0&0&0\\1&0&0\\1&0&0\end{bmatrix}$$

2. We consider S={5,7,8,4} and T={2,6,3,7} are the subsets of p. Then the incidence matrix of (S, T) is

$$E(S,T)=(e_{i,j})=\begin{bmatrix}1&0&1&0\\1&1&1&1\\1&1&1&1\\1&0&1&0\end{bmatrix}$$

3.8 Definition

If T= { $y_1, y_2 \ldots y_m$} be a S-prime Meet-closed subset of P containing S ($m \geq n$). Let D={$d_1,d_2,d_3,\ldots,d_m$ }be any subset of P containing the elements $4(x_i \wedge x_j)+1$; i,j=1,2,3,…..,n. Let the elements of D be arranged so that,

$$d_i \leq d_j \Rightarrow i \leq j$$

3.9 Theorem

If T={ $y_1, y_2 \ldots y_m$} be a S-prime Meet closed subset of P containing S={ $x_1, x_2 \ldots x_n$} , ($m \geq n$) then $(s)_f = E \wedge E^T = AA^T$

Where, E=E(S,T) ; $\Lambda = diag(g_{f,t}(y_1), g_{f,t}(y_2), \ldots, g_{f,t}(y_m))$

and $A = E\Lambda^{\frac{1}{2}}$. Ex:2

Proof:

Now we consider the Example

S={ 2,3}, T={1,2,3}

Then by using definition(2.3),

$$(s)_f = [f(4(x_i \wedge x_j)+1)] = \begin{bmatrix} f(4(2\wedge 2)+1) & f(4(2\wedge 2)+1)\\ f(4(3\wedge 2)+1) & f(4(3\wedge 3)+1)\end{bmatrix}$$

$$=\begin{bmatrix} f(4(2)+1) & f(4(2)+1)\\ f(4(2)+1) & f(4(3)+1)\end{bmatrix}$$

$$=\begin{bmatrix} f(9) & f(9)\\ f(9) & f(13)\end{bmatrix}$$

$$E=E(S,T) = \begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}$$

$$\Lambda = diag(g_{t,f}(d_1), g_{t,f}(d_2), g_{t,f}(d_3))$$

$$g_{s,f}(x_1) = f(x_1),$$

$$g_{s,f}(x_2) = f(x_2) - f(x_1),$$

$$g_{s,f}(x_3) = f(x_3) - f(x_1)$$

g(1) = f(1) = f(9)

g(2) = f(2) - f(1) = f(9) = f(3)

g(3) = f(3)-f(2) = f(13) = f(9)

$$\Lambda = diag(g_{t,f}(d_1), g_{t,f}(d_2), g_{t,f}(d_3))$$

$$E \wedge E^T = \begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}$$

$$\begin{bmatrix} f(9) & 0 & 0\\ 0 & f(9)-f(3) & 0\\ 0 & 0 & f(13)-f(9)\end{bmatrix}\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}$$

$$\begin{bmatrix} f(3) & f(9)-f(3) & 0\\ f(3) & f(9)-f(3) & f(13)-f(9)\end{bmatrix}\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}$$

$$=\begin{bmatrix} f(9) & f(9)\\ f(9) & f(13)\end{bmatrix} = (S)_f$$

Now $A = E\Lambda^{\frac{1}{2}}$

$A^T = (E\Lambda^{\frac{1}{2}})^T = (\Lambda^{\frac{1}{2}})^T E^T$

$AA^T = E \Lambda E^T$

$$=\begin{bmatrix} f(9) & f(9)\\ f(9) & f(13)\end{bmatrix}$$

$(S)_f = AA^T = E \Lambda E^T$.

Let S={1,2,3} Then

$$(S)_f = \begin{bmatrix} f(5) & f(5) & f(5)\\ f(5) & f(9) & f(5)\\ f(5) & f(5) & f(13)\end{bmatrix}$$

$diag(S)_f$ = $f(5)[f(9)\text{-}f(13)\text{-}f(5)^2]$-
$f(5)[f(5)\text{-}f(13)\text{-}f(5)^2]$+
$f(5)[f(5)^2\text{-}f(5)f(9)]$

= $f(5)f(9)f(13)-f(5)^2- f(5)^2f(13)+f(9)^2+f(13)^3-f(5)^2f(9)$

= $f(5)f(9)f(13)+f(5)^3-f(5)^2f(9)-f(9)^2f(13)$

Proof:

Now we cons ider the example,

S = {1,2 } and T = {1,2,3}. Then

$$(S)_f = [f(4(x_i \wedge x_j)+1)] = \begin{bmatrix} f(4(1\wedge 1)+1) & f(4(1\wedge 2)+1)\\ f(4(2\wedge 1)+1) & f(4(2\wedge 2)+1)\end{bmatrix}$$

$$=\begin{bmatrix} f(9) & f(5)\\ f(5) & f(9)\end{bmatrix}$$

The incidence matrix of S and T is,

$$E=E(S,T) = \begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}$$

Since $\Lambda = diag(g_{t,f}(d_1), g_{t,f}(d_2), g_{t,f}(d_3))$

and by $(g_{t,f}(x_j) = f(x_j) - f(x_{j-1})$

where j=2,3,4,…,n.

ISSN: 2278 – 7798

$$g_{s,f}(x_1) = f(x_1); \qquad g_{s,f}(x_2) = f(x_2) - f(x_1);$$
$$g_{s,f}(x_3) = f(x_3) - f(x_2).$$

$$g(1) = f(5),\ g(2) = f(9) - f(5),$$
$$g(3) = g(13) - g(9)$$

$E\,\Lambda\,E^T$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f(5) & 0 & 0 \\ 0 & f(9) - f(5) & 0 \\ 0 & 0 & f(13) - f(9) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f(9) & f(5) \\ f(5) & f(13) \end{bmatrix}$$

Now $A = E\Lambda^{\frac{1}{2}}$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f(5) & 0 & 0 \\ 0 & f(9) - f(5) & 0 \\ 0 & 0 & f(13) - f(9) \end{bmatrix}$$

$A^T = (E\,\Lambda^{\frac{1}{2}})^T = (\Lambda^{\frac{1}{2}})^T E^T$

$$= \begin{bmatrix} f(5) & 0 & 0 \\ 0 & f(9) - f(5) & 0 \\ 0 & 0 & f(13) - f(9) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} f(9) & f(5) \\ f(5) & f(13) \end{bmatrix}$$

3.10 Theorem

If S is a s-prime meet –closed. Then $\det(S)_f = \prod_{i=1}^{n} g_{s,f}(x_i)$

Proof:

The theorem is proved and verified with a suitable example. Consider the set S= {1, 2, 3}

Then

$$(S)_f = \begin{bmatrix} f(4(1\wedge 1)+1) & f(4(1\wedge 2)+1) & f(4(1\wedge 3)+1) \\ f(4(2\wedge 1)+1) & f(4(2\wedge 2)+1) & f(4(2\wedge 3)+1) \\ f(4(3\wedge 1)+1) & f(4(3\wedge 2)+1) & f(4(3\wedge 3)+1) \end{bmatrix}$$

$$= \begin{bmatrix} f(5) & f(5) & f(5) \\ f(5) & f(9) & f(5) \\ f(5) & f(5) & f(13) \end{bmatrix}$$

Let

$(S)_f$ = $f(5)[f(9)f(13)-f(5)^2]$- $f(5)[f(13)-f(5)^2]$+ $f(5)[f(5)^2 - f(5)f(9)]$

= $f(5)f(9)f(13)-f(5)^3$- $[f(5)^2f(13)]+f(5)^3$ $-f(5)^3+f(5)^2f(9)$

= $f(5)f(9)f(13)+f(5)^3-f(5)^2f(13)-f(5)^2f(9)$

By using example,

g(5) = f(5);

g(9) = f(9)-f(5)

g(13) = f(13)-f(5)

$$\prod_{i=5,9,13} \big(g(x_i)\big) = g(x_1)g(x_2)g(x_3)$$

= f (5) [f (9)-f (5)][f (13)-f(5)]

= $[f(5)\,f(9)-f(5)^2][f(13)-f(5)]$

= $f(5)\,f(9)f(13)-[f(5)^2f(13)]+f(5)^3f(5)^2f(9)$

= $f(5)\,f(9)f(13)+f(5)^{3-}f(5)^2f(13)-f(5)^2f(9)$

From the equation (1) and(2),to obtain

det $(S)_f = \prod_{i=1}^{n}\big(g_{s,f}(x_i)\big)$

Hence the theorem is proved.

3.11 Corollary

If S={$x_1, x_2, x_3, \ldots, x_n$} is a chain with $x_1 < x_2 < x_3 \ldots < x_n$. Then

det $(S)_f = f(x_1)\prod_{i=1}^{n}[f(x_1) - f(x_{i-1})]$

Proof:

By using theorem,

If S is a S-prime meet –closed then

det $(S)_f = \prod_{i=1}^{n}\big(g_{s,f}(x_i)\big)$ and the result ,

$f(5)\,f(9)f(13)+f(5)^{3-}$-$f(5)^2f(13)-f(5)^2f(9)$

We have , f (5) [f (9)-f (5)][f(13)-f(5)]

det $(S)_f$ = g(1)g(2)g(3)

3.12 Theorem

Let T ={ $y_1, y_2, y_3, \ldots, y_m$ } be a S-prime Meet –closed subset of P containing S={ $x_1, x_2, x_3, \ldots, x_n$ }. Then,

det $(S)_f$

$$= \sum_{1\le k_1 \le \ldots \le k_n \le m} \det[E(k_1, k_2, \ldots, k_n)^2\, g_{T,f}(y)_{k1},, g_{T,f}(y)_{k2},,\ldots g_{T,f}(y)_{kn}$$

Where, E=E(S,T)

Proof:

$(S)_f = E\,\Lambda\,E^T$, also det(E) = det(E^T), by using known theorem.

Now we consider the example ,

S = {2,3} and T ={1,2,3} .Then,

$$(S)_f = [f(4(x_i \wedge x_j)+1)] = \begin{bmatrix} f(4(2\wedge 2)+1) & f(4(2\wedge 3)+1) \\ f(4(3\wedge 2)+1) & f(4(3\wedge 3)+1) \end{bmatrix}$$

$$(S)_f = \begin{bmatrix} f(9) & f(5) \\ f(5) & f(13) \end{bmatrix}$$

The incident matrix of S&T is,

$$E = E(S,T) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

ISSN: 2278 – 7798

$E \wedge E^T=$

$$\begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}f(5)&0&0\\0&f(9)-f(5)&0\\0&0&f(13)-f(5)\end{bmatrix}\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}f(9)&f(5)\\f(5)&f(13)\end{bmatrix}$$

$(S)_f = E \wedge E^T$

Also, det (E) $\Rightarrow E=\begin{vmatrix}1&1&0\\1&1&1\end{vmatrix}=0$

det (E^T) $\Rightarrow \begin{vmatrix}1&1\\1&1\\0&1\end{vmatrix}=0$

det (E) = det (E^T)

det $(S)_f$

$$=\sum_{1\le k_1\le..\le k_n\le m}\det[E(k_1,k_2,...,k_n)^2 g_{T,f}(y)_{k1},,g_{T,f}(y)_{k2},,...g_{T,f}(y)_{kn}]$$

Hence proved.

4. Determinant and inverse S-prime Meet Matrix on posets

4.1 Definition

Let the element of S be arranged so that $x_i \le x_j \Rightarrow i<j$. Let $d=\{d_{1,}d_{2,}d_{3,},...,d_{n,}\}$ be any subset of P continuous on element $x_i \wedge x_j, i=1,2,3,...,n$. let an elements of D be arranged so that $d_i \le d_j \Rightarrow i \le j$. The arithmetical function $g_{D,f}$ on D derived by

$$g_{D,f}(d_k)=\sum_{d_v\le d_k} f(d_k)\mu_D(d_v,d_k)$$

where $\mu(D)$ is the mobius function of the posets (D, $\le$).

4.2 Theorem

Let S= $\{x_1, x_2, x_{3,...,} x_n\}_{be}$ subset of P with D= $\{d_1, d_2, d_{3,...,} d_m\}$. Let g be an arithmetical function then $(S)_f$ =E dig(g(d_1), g(d_2), g(d_3),…, g(d_m)) E ^{T} where E=(S,D)

Proof:

Similar to a proof of 3.9

4.3 Theorem

Let S, D, F, and g be as in the theorem 4.2 then where det $(S)_f$

$$=\sum_{1\le k_1\le..\le k_n\le m}\det[E_{(k_1,k_2,...k_n)}{}^2 g(d_1).g(d_2).g(d_3)....g(d_m)]$$

Where E $_{(k_1, k_2, k_{3,...,} k_n)}$ is the sub-matrix of E, E(S,D) consist of the $k_1^{th}, k_2^{th}, …, k_n^{th}$ columns of E . Further if g is a function with positive value then det (M) $\ge g(s_1)g(s_2)...g(s_m)$ and the equality holds iff S is meet-closed .

Proof:

Since $(S)_f = E(g(d_i))E^T$ and $\det E = \det E^T$ so the proof of the theorem is obvious.

4.4 Theorem

If S is lower closed subset of P. then

$\det(S)_f = \prod_{i=1}^{n}(g(d_i))$ where,

$$g(d_i)=\sum_{d_j\le d_i}(4d_i+1)\mu(d_j,d_i)$$

Proof:

By using theorem 4.2 and definition 4.1 to get the proof.

4.5 Theorem

$B=(b_{ij})$ where

$$b_{ij}=\frac{(-1)^{i+j}}{\det(S)_f}\sum_{1\le k_1\le....\le k_n}\det E(S_j)_{(k_1,k_2,.........k_{n-1})}$$

$$\det E(S_i)_{(k_1,k_2,.........k_{n-1})} X g(d_1)g(d_2)......g(d_n)$$

Proof:

By using the theorem 4.2, 4.3.

4.6 Theorem

If $(S)_f=(f_{ij})$ is invertible then the inverse of $(S)_f$ in the nxn matrix $B=(b_{ij})$ where $b_{ij}=\frac{\alpha_{ji}}{\det(s)_f}$, Where α_{ji} in the co-factor of the ji^{th} entry of $(S)_f=(f_{ij})$.

Proof :

It is a general method used to prove.

4.7 Theorem

Suppose that S is meet-closed .if $(S)_f$ is invertible then the inverse of $(S)_f$ is $B=(b_{ij})$

Where $b_{ij}=\sum_{\substack{d_i\le d_k\\ d_j\le d_k}}\frac{\mu(d_i,d_k)\mu(d_j,d_k)}{g(d_k)}$

Proof :

It is similar to the proof of theorem.

4.8 Example

Let S= {1, 2} Then by definition(2.5)

$(S)_M = \lfloor 4(x_i \wedge x_j)+1 \rfloor$

$$(S)_f=\begin{bmatrix}(4(1\wedge 1)+1)&(4(1\wedge 2)+1)\\(4(2\wedge 1)+1)&(4(2\wedge 2)+1)\end{bmatrix}$$

$$=\begin{bmatrix}4(1)+1&4(1)+1\\4(1)+1&4(2)+1\end{bmatrix}$$

$$=\begin{bmatrix}5&5\\5&9\end{bmatrix}$$

And by definition(4.4) Det $(s)_M = \prod_{i=1}^{n}\left(g(d_i)\right)$

$$g(d_i) = \sum_{d_j \le d_i}(4d_i+1)\mu(d_j,d_i)$$

$$g(1) = \sum_{d_j \le 1}(4d_i+1)\mu(d_j,1) = (4(1)+1)\mu(1,1) = 5$$

$$g(2) = \sum_{d_j \le 2}(4d_i+1)\mu(d_j,2)$$

$$= (4(1)+1)\mu(1,2) + (4(2)+1)\mu(2,2)$$

$$= 5(-1) + 9(1) = 4$$

det $(S)_M = g(1)g(2) = (5)(4) = 20$

And by using definition (4.7),

$$b_{11} = \sum_{\substack{1\le d_k \\ 1\le d_k}} \frac{\mu(1,d_k)\mu(1,d_k)}{g(d_k)}$$

$$b_{11} = \sum_{\substack{1\le d_k \\ 1\le d_k}} \frac{\mu(1,d_k)^2}{g(d_k)}$$

$$b_{11} = \frac{\mu(1,1)^2}{g(1)} + \frac{\mu(1,2)^2}{g(2)}$$

$$b_{11} = \frac{1}{5}+\frac{1}{4}\quad\frac{9}{20}$$

$$b_{12} = \sum_{\substack{1\le d_k \\ 2\le d_k}} \frac{\mu(1,d_k)\mu(2,d_k)}{g(d_k)}$$

$$b_{12} = \frac{\mu(1,2)\mu(2,2)}{g(2)}$$

$$b_{12} = \frac{(1)(-1)}{4}$$

$$b_{12} = \frac{-1}{4}$$

$$b_{21} = \sum_{\substack{2\le d_k \\ 1\le d_k}} \frac{\mu(2,d_k)\mu(1,d_k)}{g(d_k)}$$

$$b_{21} = \frac{\mu(2,1)\mu(2,2)}{g(2)}$$

$$b_{21} = \frac{-1}{4}$$

$$b_{22} = \sum_{\substack{2\le d_k \\ 2\le d_k}} \frac{\mu(2,d_k)\mu(2,d_k)}{g(d_k)}$$

$$b_{22} = \frac{1}{4}$$

$$(S)^{-1}_M = \begin{bmatrix} \frac{9}{20} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$$

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ISSN: 2278 – 7798