DETOUR DOMINATION NUMBER OF A GRAPH

A.NELLAI MURUGAN[1]A.ESAKKIMUTHU[2]AND G.MAHADEVAN[3]

***Abstract*—In [3], the author introduced the *detour number of a graph* . In this paper, we introduce the new concepts of detour dominating set of graphs. A subset S of vertices in a graph G is a called a *detour dominating set*if S is both a detourset and a dominating set. The detourdomination number $\gamma_{dn}(G)$is the minimum cardinality of *a detour dominating set.* Any detourdomination of cardinality $\gamma_{dn}(G)$is called $\gamma_{dn}(G)-set$ of G. In this paper, we study detourdomination on graphs.**

***Keywords* — Domination, detour distance, detour number, detour dominating set, detourdomination number.**
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I INTRODUCTION

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by *V(G)* and the edge set by *E(G).* We define the order of *G* by $n = n(G) = |V(G)|$ and the size by $m = m(G) = |E\ (G)|$. For a vertex$v \in V(G)$, the *open neighborhood N(v)* is the set of all vertices adjacent to *v*, and *N[v] = N(v)*∪*{v}* is the *closed neighborhood* of *v*. The *degree d(v)* of a vertex *v* is defined by *d(v) = |N(v)|*. The *minimum* and *maximum degrees of a graph G are* denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

For $X \subseteq V(G)$let *G[X]* the sub graph of *G* induced by *X*, *N(X)* = $\cup_{x\in X}$*N(x)* and *N[X]*=$\cup_{x\in X}$*N(x).*

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A.NellaiMurugan, *Dr.S.S.Pillai Centre for Research in Mathematics, V. O. Chidambaram college Thoothukudi-628 008,Tamilnadu, India*

A.Esakkimuthu, *Full Time Research Scholar, Dr.S.S.Pillai Centre for Research in Mathematics, V. O. Chidambaram college, Thoothukudi-628008, Tamilnadu, India*

G.Mahadevan, *Department of Mathematics Gandhigram Rural Institute- Deemed University, Gandhigram, Dindigul District, TamilNadu, India*

If *G* is a connected graph, then the distance *d(x, y)* is the length of a shortest $x-y$ path in *G*. The diameter $diam(G)$ of a connected graph is defined by *diam(G)* = $max_{x,y\in V(G)}d(x,y)$. An $x-y$ path of length *d(x, y)* is called an $x-y$ *geodesic*. A vertex *v* is said to lie on an $x-y$ geodesic *P* if *v* is an internal vertex of *P*. The closed interval *I[x, y]* consists of *x, y* and all vertices lying on some $x-y$ geodesic of *G*, while for $S \subseteq V(G)$, *I[S]* = $\bigcup_{x,y\in S} I[x,y]$. If *G* is a connected graph, then a set *S* of vertices is a geodetic *set* if *I[S]* = *V(G).* The minimum cardinality of a geodetic set is the geodetic *number* of *G*, and is denoted by *g(G).* The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a $g(G)$ *−set*. [1, 2, 3, 4, 6].

For vertices x and y in a connected graph$G$, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in$G$. For any vertex $u$of$G$, the *detour eccentricity* of $u$is$e_D(u) = \max\{D(u,v): v \in V\}$. A vertex v of G such that $D(u,v) = e_D(u)$ is called a *detour eccentric vertex* of u.The*detour radiusR* and *detour diameterD* of $G$are defined by $R = rad_D G = \min\{e(v): v \in V\}$ and $D = diam_D G = \max\{e(v): v \in V\}$ respectively. An$x-y$ path of length $D(x,y)$ is called an $x-y$*detour*. The *closed interval* $I_D[x,y]$ consists of all vertices lying on some $x-y$ detour of G, while for$S \subseteq V, I_D[S] = \bigcup_{x,y\in S} I_D[x,y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [3]

A vertex in a graphdominates itself and its neighbors. A set of vertices *S* in a graph *G* is a *dominating set* if *N*[*S*] = *V* (*G*). The *domination number* $\gamma(G)$ of *G* is the minimum cardinality of a dominating set of *G*. The domination number was introduced in [6].

It is easily seen that a dominating set is not in general a detourset in a graph G. Also the converse is not valid in general. This has motivated us to study the new domination conception of detourdomination.

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We investigate those subsets of vertices of a graph that are both a detourset and a dominating set. We call these sets detour dominating sets. We call the minimum cardinality of a detour dominating set of G, the detour domination number of G. It is easily seen that every extreme vertex belongs to every detour dominating set.

1.1 Remark:[5]

1) A dominating set D of G is a minimal dominating set of G if and only if for every v ∈ D, there exists at least one vertex w ∈ V − (D − {v}) which is not dominated by D − {v}.
2) A dominating set D of G is a minimal dominating set of G if and only if for every v ∈ D, there exists at least one vertex w ∈ V − (D − {v}) such that N[w] ∩ D = {v}.

1.2 Theorem:[3] Every end-vertex of a nontrivial connected graph G belongs to every detour set of G

1.4 Theorem [3] If T is atree with k end-vertex, then $dn(T) = k$

1.5 Theorem [3] For every connected graph G, $rad_D(G) \leq diam_D(G) \leq 2\ rad_D(G)$

1.6 Definition: Two vertices u and v of a graphs G are said to be antipodal vertices G if $d(u,v) = diam\ G$.

II.Detour domination Number of Graph

Definition 2.1: Let $G = (V,E)$ be any connected graph with at least two vertices. A detour dominating set of G is a subset S of $V(G)$ which is both dominating and detourset of G.

A detour dominating set S is said to be a minimal detour dominating set of G if no proper subset of S is a detour dominating set of G. A detour dominating set S is a said to be a minimum detour dominating set of G if there exists no detour dominating set S' such that $|S'| < |S|$. The cardinality of a minimum detour dominating set of G is called the detour domination number of G. It is denoted by $\gamma_{dn}(G)$. Any detour dominating set S of G of cardinality γ_{dn} is called a $\gamma_{dn} -$ set of G.

Remark 2.2: Let G be a connected graph with $p \geq 2$ vertices. Then, $\gamma(G) \leq \gamma_{dn}(G)$. Strict inequality is also true in the above relation. For example, consider P_{15}. Let $V(P_{15})$ = $\{ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. $\gamma(P_{15}) = 5$,Since $\{v_2, v_5, v_8, v_{11}, v_{14}\}$ is a $\gamma - set.$ $\gamma_{dn}(P_{15}) = 6$,since$\{v_1, v_4, v_7, v_{10}, v_{13}, v_{15}\}$ is a $\gamma_{dn} - set\ of\ P_{15}$.

Example 2.3: Considering the graph G in figure 2.3(a),

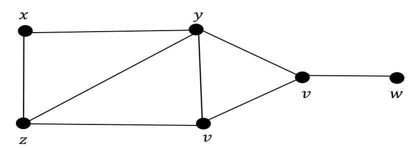

Figure2.3(a)

$\{y, w\}$is the dn -set of G. So $dn(G) = 2$.
$\{y, w\}$is also the $\gamma_{dn} - set$ of G. Therefore, $\gamma_{dn}(G) = 2$
Also $\gamma(G) = 2$, though G has more than one $\gamma - set$.
In this example, $dn(G) = \gamma_{dn}(G) = \gamma(G)$.

Remark 2.4:
In general, $\gamma_{dn}(G), dn(G)$and $\gamma(G)$, all need not be equal.
For example consider P_8.
$\gamma_{dn}(P_8) = \{ v_1, v_4, v_6, v_8\} = 4, dn(P_8) = \{ v_1, v_8\} = 2$and $\gamma(P_8) = \{v_2, v_5, v_8\} = 3$.
Further, $\gamma_{dn}(P_7) = 3 = \gamma(P_7)$, But $dn(P_7) = 2$.
Also, $\gamma_{dn}(P_3) = dn(P_3) = 2$, whereas, $\gamma(P_3) = 1$.

Remark 2.5: Let G = (V,E) be any connected graph with at least two vertices. Then,

1. $\gamma_{dn}(G) \geq dn(G)and\gamma_{dn}(G) \geq \gamma(G)$.
2. Every detour dominating set of G contains all pendant vertices of G.
3. If G is a graph with at least one pendent vertex, then for every detour dominating Set D of G, V-D is not a detour dominating set of G.
4. Every super set of a detour dominating set of G is a detour dominating setof G.

Proposition 2.6: For a star graph G, then $\gamma_{dn}(G) = p - 1$.

Proof: Let G = $K_{1,n}$ with $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Let S be a minimum detour dominating set of $K_{1,n}$. By Remark 2.5$\{v_1, v_2, v_3, \dots\dots. v_n\} \subseteq S$.Since $\{v_1, v_2, v_3, \dots\dots. v_n\}$ itself is a detour dominating setof $K_{1,n}$, S = $\{v_1, v_2, v_3, \dots\dots, v_n\}$. Therefore, $\gamma_{dn}(G) = n = p - 1$.

Proposition 2.7: If G is a bi-star graph G, then $\gamma_{dn}(G) = p - 2$.

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Proof: Let G $= B(r, s)$where$r, s \geq 1$. Suppose $V(B(r, s)) = \{u, v, u_i, v_i : 1 \leq i \leq r \ and \ 1 \leq j \leq s\}$and $E(B(r, s)) = \{uv, uu_i, vv_i : 1 \leq i \leq r \ and \ 1 \leq j \leq s\}$. Let S be a minimum detour dominating set of $B(r, s)$. By Remark 2.5.$\{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\} \subseteq S$. Since $\{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\}$ is itself a detour dominating set of G,$S = \{u_1, u_2, \ldots \ldots . u_r, v_1, v_2, \ldots \ldots . v_s\}$. Therefore, $\gamma_{dn}(B(r, s)) = p - 2$.

Theorem 2.8:

$$\gamma_{dn}(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if \ n \geq 5 \\ 2 & if \ n = 2,3 \ or \ 4 \end{cases}$$

Proof. Let $P_n = \{v_1, v_2, v_3, \ldots \ldots . v_n\}$. If n = 2,3,or 4, then $\{v_1, v_n\}$ is a minimum detour dominating set of P_n.therefore, $\gamma_{dn}(P_2) = \gamma_{dn}(P_3) = \gamma_{dn}(P_4) = 2$. Let $n \geq 5$.We observe that every detour dominating set of P_n is a dominating set containing the end vertices of P_n. Let D_1be a minimum dominating set containingv_1, v_n. Therefore, $|D_1| \leq |S|$. As D_1 is also a detour dominating setof G,$|S| \leq |D_1|$. So, we have, $\gamma_{dn}(P_n) = |S| = |D_1|$. Let D be a minimum dominating set of P_n.then,

$$|D_1| = \begin{cases} |D| = \left\lceil \frac{n}{3} \right\rceil & if \ n \equiv 1 \ (mod \ 3) \\ |D| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

$=\left\lceil \frac{n-4}{3} \right\rceil + 2$. (By Remark 2.9)

Therefore,$\gamma_{dn}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Remark 2.9:

$$\left\lceil \frac{n-4}{3} \right\rceil + 2 = a \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if \ n \equiv 1 \ (mod \ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

Theorem 2.10:

$$\gamma_{dn}(C_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if \ n \geq 5 \\ 2 & if \ n = 3 \ or \ 4 \end{cases}$$

Proof. Let $C_n = \{v_1, v_2, v_3, \ldots \ldots . v_n\}$. If n = 3, or 4, then $\{v_1, v_n\}$ is a minimum detour dominating set of P_n.therefore, $\gamma_{dn}(C_3) = \gamma_{dn}(C_4) = 2$. Let $n \geq 5$. By theorem 2.8, $C_n \equiv P_n$

Hence,$\gamma_{dn}(C_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if \ n \geq 5 \\ 2 & if \ n = 3 \ or \ 4 \end{cases}$

Theorem 2.11: Let$W_n = C_{n-1} + K_1$, $n \geq 5$ denoted the wheel graph on n vertices. Then, $\gamma_{dn}(W_n) = 2$.

Proof. Let$V(W_n) = \{v, v_1, v_2, \ldots \ldots . ., v_{n-1}\}$ with v as its central vertex. Let $S = \{v, v_{n-1}\}$be a minimum detour dominating set of W_n. Compulsory central vertex v and otherwise any one vertex v_{n-1}. Distance of (v, v_{n-1}) is longest path of $v, v_1, v_2, \ldots \ldots . ., v_{n-1}$. $v$is adjacent to all other vertices$v_1, v_2, \ldots \ldots . ., v_{n-1}$. This is satisfying detour dominating set condition.

Hence,$\gamma_{dn}(W_n) = 2$

Theorem 2.12: $\gamma_{dn}(K_{m,n}) = 2$ if $m = n = 1 \ and \ m, n \geq 2$.

Proof. Let$G = K_{m,n}$ with the bipartition $V_1 = \{a_1, a_2, a_3, \ldots \ldots ., a_m\}$and $V_2 = \{b_1, b_2, b_3, \ldots \ldots ., b_n\}$. If $= n = 1 \ then G = K_2$. Cleary, $\gamma_{dn}(G) = 2$.If $, n \geq 2$. Let $S = \{a_1, b_1\}$ be a minimum detour dominating set $K_{m,n}$. Distance of (a_1, b_1) is longest path of $(a_1, b_2, a_2, b_1), (a_1, b_3, a_3, b_1), (a_1, b_4, a_4, b_1), \ldots \ldots \ldots \ldots$

$\ldots \ldots . (a_1, b_n, a_m, b_1)$. This is a detour set, and (a_1, b_1) is dominating set of $K_{m,n}$. This is satisfying detour dominating set condition.

Hence, $\gamma_{dn}(K_{m,n}) = 2$.

Theorem2.13: For a complete Graph G. Then $\gamma_{dn}(G) = 2$

Proof. Let$G = K_n$. $V(G) = \{v_i ; 1 \leq i \leq n\}$. Any two vertices longest path of $(v_1, v_n)$is $v_1, v_2, v_3, \ldots \ldots ., v_n$. And v_1 is adjacent to $v_i ; 2 \leq i \leq n$. This is satisfying detour dominating set condition.

Hence, $\gamma_{dn}(K_n) = 2$.

Theorem 2.14: $2 \leq \gamma_{dn}(G) \leq p - 1$

Proof: By Remark 2.5 and by theorem 1.3, $\gamma_{dn}(G) \geq 2$. Further, any detour dominating set is a subset of V(G) implies $\gamma_{dn}(G) \leq p - 1$.

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In the above proposition, upper bound is sharp, as $\gamma_{dn}(K_{1,p-1}) = p-1$ and the lower bound is sharp (see example 2.16(a))

Theorem 2.15: Let G be any graph with k support vertices and l end vertices. Then,

$l \leq \gamma_{dn}(G) \leq p-k.$

Proof: LetL and K denote the set of all end and support vertices of G respectively and$|L| = l$; $|K| = k$. Clearly, $l \geq k$. By Remark 2.5, L is a subset of everydetour dominating set of G. So, $\gamma_{dn}(G) \geq l$. Further every vertex of K lies in a geodesic joining two vertices of L as well as dominated by the vertices of L. Therefore, it is clear that $V-K$ is a detour dominating set of G and so $\gamma_{dn}(G) \leq |V-K| = |V| - |K| = p-k$. Hence the proof.

Corollary 2.16: Let T be any tree with k support vertices and l end vertices such that$l+k=p$. Then, $\gamma_{dn}(G) = p-k$.

The following example shows even if$l+k < p$, $\gamma_{DG}(G) = p-k$.

Example2.17: consider the graph G in figure 2.17(a)

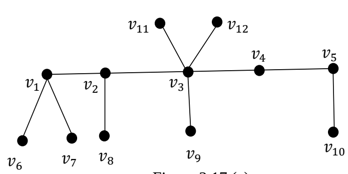

Figure 2.17 (a)

Clearly, $S = \{v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is a$\gamma_{dn}(G) = 8 = |V-K|$.

Proposition 2.18: Let $G = (V, E)$ be a connected graph of order p and diameter d. Then, $\gamma_{dn}(G) \leq p - d + \left\lceil \frac{d}{3} \right\rceil$.

Proof. Let $P: (u = v_0, v_1, \ldots\ldots\ldots, v_d = v)$be a path of length d in G. If $= \{v_1, v_2, \ldots\ldots\ldots, v_{d-1}\}$, then $V-S$ is a detour set of G. Also, it is a dominating set of $< V - (S - \{v_1, v_{d-1}\}) >$. Let $P' = (v_2, v_3, \ldots\ldots\ldots, v_{d-2})$. $|V(P')| = d-3$. Let D' be a minimum dominating set of P'. By Theorem 1.1$|D'| = \left\lceil \frac{d-3}{3} \right\rceil$. Let $D = (V-S) \cup D'$. Clearly, D is a detour dominating set of G. Therefore, $\gamma_{dn}(G) \leq |D| = |(V-S) \cup D'| \leq p - d + 1 + \left\lceil \frac{d-3}{3} \right\rceil = p - d + 1 + \left\lceil \frac{d}{3} \right\rceil - 1 = p - d + \left\lceil \frac{d}{3} \right\rceil$.

Remark 2.19: Equality holds in the above theorem. For example, considering the graph G as in Figure 2.19(a)

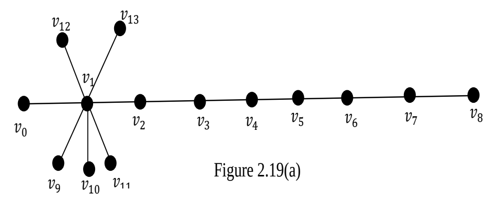

Figure 2.19(a)

$p = 14. d = 8$and so $p - d + \left\lceil \frac{d}{3} \right\rceil = 9$.

$S = \{v_0, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_3, v_5, v_8\}$is a minimum detour dominating set of G and $\gamma_{dn}(G) = |S| = 9 = p - d + \left\lceil \frac{d}{3} \right\rceil$.

Theorem 2.20: Let G be a connected graph of order $p(\geq 2)$ and diameter d. Suppose $\delta \geq 3$. Then, $\gamma_{dn}(G) \leq p - d + 1$.

Proof: Let $u, v \in V(G)$.As diam (G)=d, There exists a shortest path P: $(u = v_0, v_1, \ldots\ldots, v_d = v)$ of length d in G. If S=$\{v_1, v_2, \ldots\ldots, v_{d-1}\}$, then $V-S$ is a detourset of G. since $\delta \geq 3$, each vertex of S is adjacent to atleast one vertex of $V-S$ is a detour dominating set of G and so $\gamma_{dn}(G) \leq |V-S| = p - (d-1) = p - d + 1$.

Observation 2.21: Let G be connected graph with $p(\geq 2)$ vertices. Then

1. A minimal detourset of G which is also a dominating set of G is a minimal detour dominating set of G.

2. A minimum detourset (or dn-set) of G which is also a dominating set of G is a minimum detour dominating set of G. ($or \quad \gamma_{dn} -$ set) of G.

3. Any minimal dominating set of G which is also a detourset of G is a minimal detour dominating set of G.

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4. Any minimum dominating set (or γ-set) of G which is also a detourset of G is a minimum detour dominating set of G. ($or \quad \gamma_{dn} -$set) of G.

III. FURTHER RESULTS ON DETOUR DOMINATION NUMBER OF GRAPHS

Theorem 3.1: Let G = (V,E) be any graph. Suppose S is a proper subset of V (G) such that the sub graph of G induced by S is complete, then S is not a detour dominating set of G.

Proof: Let $v \in V - S$. Since the sub graph of G induced by S is complete, d(x,y)=1 for every$x, y \in S$. Therefore, v does not lie on any detour joining any pair of vertices of S. Hence, S is not a detour dominating set of G.

Theorem 3.2: Let G= (V, E) be a graph. Then, every detour dominating set of G contains all the extreme vertices of G.

Proof: Let S be the set of all extreme vertices of G. By 1.3, every detourset of G contains S. As every detour dominating set of G is a detourset of G, every detour dominating set of G contains S.

Theorem 3.3: Let G = (V, E) be any graph. If S, the set of all extreme vertices of G, is a detour dominating set of G, then S is the unique minimum detour dominating set of G.

Proof: Suppose S is a detour dominating set of G. Then, $\gamma_{dn}(G) \leq |S|$.By Theorem 3.2, $\gamma_{dn}(G) \geq |S|$. Therefore, $\gamma_{dn}(G) = |S|$ and so S is a minimum detour dominating set of G. Again by Theorem 3.2, every minimum detour dominating set of G contains S and so S is the unique minimum detour dominating set of G.

Remark 3.4: If S, the set of all end vertices of G, is a detour dominating set of G, then S is the unique minimum detour dominating set of G.

Theorem 3.5: For every positive integer $k \geq 2$, there exists a graph G with $\gamma_{dn}(G) = k$.

Proof: Let $k \geq 2$. Consider the complete bipartite graph$K_{1,k}$. Let $S = V(K_{1,k}) - \{v\}$ where v is the central vertex. Obviously, S is a detour dominating set of G. Therefore, by Remark 3.4, S is the unique minimum detour dominating set of $K_{1,k}$ and so $\gamma_{dn}(K_{1,k}) = k$.

Theorem 3.6: For every pair k, p of integer such that $2 \leq k \leq p$, there exists a connected graph G of order p such that $\gamma_{dn}(G) = k$.

Proof: As $\gamma_{dn}(K_2) = 2$, the result is true when $G \cong P_3$ is a graph as in figure 3.6(a) . Here,$\{u, v\}$ is a minimum detour dominating set of G and so $\gamma_{dn}(G) = 2$.

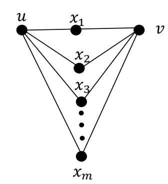

Figure 3.6(a)

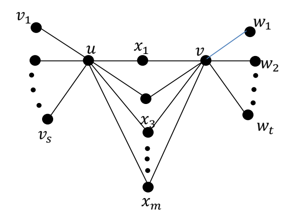

Figure 3.6(b)

Let $2 < k < p$. Consider the graph as in figure 3.6(b).

Where $m = p - (k + 1)$.Here, u and v are vertices of G such that there are exactly m paths of length 2 connecting u and v. Let $x_1, x_2, x_3, \dots\dots.. x_m$ be the internal vertices of these paths. Let s and t be positive integers such that $1 \leq s, t \leq k - 2$and $s + t = k - 1$. Attach s end vertices $v_1, v_2, v_3, \dots\dots.. v_s$ to u and t end vertices $w_1, w_2, w_3, \dots\dots.. w_t$ to v. Then,

number of vertices of $G = s + t + 2 + m = k - 1 + 2 + p - k - 1 = p$.Further, $\{v_1, v_2, v_3, \dots\dots, v_s, w_1, w_2, w_3, \dots\dots.. w_t, u\}$ is a minimum detour dominating set of G and so $\gamma_{dn}(G) = k$.

IV. GRAPHS WITH PRESCRIBED DETOUR DOMINATION NUMBER.

Theorem 4.1: Let G be a (p, q) graph and $k \geq 3$. If $dn(G) = p - k$, then $\gamma_{dn}(G) \leq p - 2$.

Proof. Let S be a detour dominating set of G. Then, $|V - S| = k$. As S is detour, every vertex of $V - S$lies in a detour connected two vertices of S of length atmost $k + 1$. So, atleast two vertices of $V - S$ are

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dominated by the vertices of S. Now, the remaining $k-2$ vertices of $V-S$ together with S constitute a detour dominating set of G. Therefore, $\gamma_{dn}(G) \leq p-k+k-2 = p-2$. Hence, $\gamma_{dn}(G) \leq p-2$.

Theorem 4.2: If $dn(G) = p-3$, then $\gamma_{dn}(G) \leq p-3$.

Proof. The result follows form Remark 2.5 and Theorem 4.1.

Theorem 4.3: Let G be a connected graph on $p(\geq 2)$ vertices. Then, $\gamma_{dn}(G) + \chi(G) = p+2$ if and only if G is complete.

Proof: If $\gamma_{dn}(G) + \chi(G) = p+2$, then $\gamma_{dn}(G) = 2 \ \ and \ \ \chi(G) = p$. Hence, G is complete. Converse is obvious.

V. RESULTS CONNECTING DOMINATION, DIAMETER, DETOUR AND DETOUR DOMINATION NUMBER OF A GRAPHS

Lemma 5.1: Given a positive integer $k \geq 2$, there exists a graph G with $\gamma(G) = dn(G) = \gamma_{dn}(G) = k$.

Proof. Consider the graph G in figure 5.1(a) ,

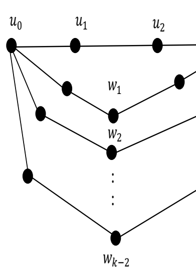

Figure 5.1(a)

Clearly, $\{u_0, u_3, w_1, w_2, \ldots, w_{k-2}\}$ is a γ –set, dn-set and γ_{dn}-set of G and so $\gamma(G) = dn(G) = \gamma_{dn}(G) = k$.

Lemma 5.2: Given two positive integer a and b with b >a , there exists a graph G such that $\gamma(G) = a \ and \ \gamma_{dn}(G) = dn(G) = b$.

Proof: Let a=1 and b>a. The star graph $K_{1,b}$ satisfies the required condition.

Let $a = 2$ and $b > a$. Let s and t be two positive integers such that s+ t = b. Considering G in figure 5.1(a)

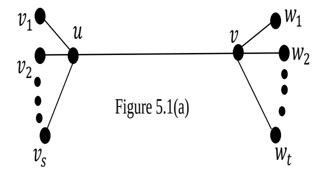

Figure 5.1(a)

$\{u, v\}$is a minimum dominating set of G and so $\gamma(G) = 2$. Further, $\{v_1, v_2, \ldots \ldots, v_s, w_1, w_2, \ldots, w_t\}$ is a minimum detourand minimum detour dominating set of G and hence $\gamma_{dn}(G) = dn(G) = s+t = b$.

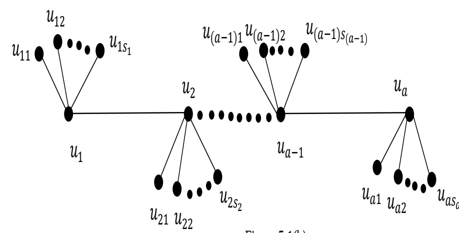

Figure 5.1(b)

Let a>2 and b>a. Let $P_a: (u_1, u_2, \ldots \ldots, u_a)$. Choose positive integer $s_1, s_2, \ldots \ldots, s_a$ such that $s_1 + s_2 + \cdots + s_a = b$. Join $s_1, s_2, \ldots \ldots, s_a$ pendant vertices $u_1, u_2, \ldots \ldots, u_a$ of P_a.The resulting graph appears as in figure 5.1(b) . Clearly, $\{u_1, u_2, \ldots \ldots, u_a\}$ is a minimum dominating set of G and so $\gamma(G) = a$ and $\{u_{11}, u_{12}, \ldots \ldots, u_{1s_1}, u_{21}, u_{22}, \ldots \ldots, u_{2s_2}, \ldots \ldots,$

$u_{a1}, u_{a2}, \ldots \ldots, u_{as_a}\}$is a $dn-set$ and $\gamma_{dn}-set$ of G and $\gamma_{dn}(G) = dn(G) = b$.

Remark5.3: Form remark 2.5,we observe the following result. Given 2 positive integer a, b with $a > 2$ and$b < a$, there exists no graph with$\gamma(G) = a$ and $\gamma_{dn}(G) = dn(G) = b$.

Theorem 5.4: Given 2 positive integer$a, b \ with 2 \leq b < a$, there exists a graph G with$\gamma(G) = \gamma_{dn}(G) = a$ and $dn(G) = b$.

Proof. For b=2,P_{3a-2} satisfies the required condition since

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1. The set of end vertices of P_{3a-2} is the unique minimum detour set of P_{3a-2} and so $dn(P_{3a-2}) = 2 = b$.

2. By Theorem 1.1 $\gamma(P_{3a-2}) = \left\lceil \frac{3a-2}{3} \right\rceil = a$.

3. Further, $3a-2 \geq 5$ as $a > 2$. Therefore, by theorem 2.8, $\gamma_{dn}(P_{3a-2}) = \left\lceil \frac{(3a-2)-4}{3} \right\rceil + 2 = \left\lceil \frac{3a-6}{3} \right\rceil + 2 = a-2+2 = a$.

Let
$b > 2$, Let $P_{3a-2b+1} = (v_1, v_2, v_3, \dots \dots, v_{3a-2b+1})$. As $> b$, $3a-2b+1 > b$. Let G be a graph as in the figure 5.4(a) which is obtained by attaching $b-1$ pendent vertices $w_1, w_2, w_3, \dots \dots, w_{b-1}$ respectively to $v_1, v_2, v_3, \dots \dots, v_{b-1}$ of the path $P_{3a-2b+1}$.

$\{w_1, w_2, w_3, \dots \dots, w_{b-1}, v_{3a-2b+1}\}$ is a minimum detour set of G and so $dn(G) = b$. The set $S = \{v_1, v_2, v_3, \dots \dots, v_{b-1}\}$ together with any minimum dominating set of the path $P' = (v_{b+1}, v_{b+2}, v_{b+3}, \dots \dots, v_{3a-2b+1})$ forms a minimum dominating set of G.

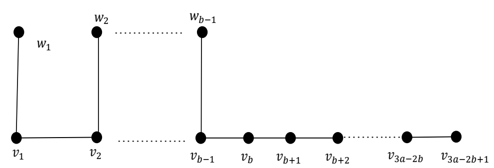

Figure 5.4(a)

Number of vertices in $P' = 3a-2b+1-b = 3(a-b)+1$ and so by theorem 1.1, $\gamma(G) = b-1+\gamma(P') = b-1+\left\lceil \frac{3(a-b)+1}{3} \right\rceil = b-1+a-b+1 = a$.

Further, $\{w_1, w_2, w_3, \dots \dots, w_{b-1}, v_{3a-2b+1}\}$ along with any minimum dominating set of the path $P'' = (v_b, v_{b+1}, v_{b+2}, \dots \dots, v_{3a-2b-1})$ forms a minimum detour dominating set of G.

Number of vertices in $P'' = 3a-2b-1-(b-1) = 3(a-b)$

Therefore, $\gamma_{dn}(G) = b + \gamma(P'') = b + \left\lceil \frac{3(a-b)}{3} \right\rceil = b + (a-b) = a$.

Remark 5.5: As in Remark 5.3, for any 2 positive integer a, b $with$ $a < b$, therefore exists no graph G with $dn(G) = b$ and $\gamma(G) = \gamma_{dn}(G) = a$

Theorem 5.6: Let G be a non-complete connected graph with $p\ (\geq 4)$ vertices. If $diam_D \geq 3$, $then$ $\gamma_{dn}(G) \leq p-2$.

Proof: Let u, v be a pair of antipodal vertices of G and let P be a path of length d between u and v, where $d = diam_D$. If u' and v' are the vertices adjacent to u and v respectively in P, then $V - \{u', v'\}$ is a detour dominating set of G. Therefore, $\gamma_{dn}(G) \leq p-2$.

Theorem 5.7: Let G be a non-complete connected Graph with $p \geq 3$ vertices. Then, $\gamma_{dn}(G) = 2$ if and only if G satisfies the following two conditions.

$$(i) diam_D < 4$$

(ii) G contains 2 antipodal vertices u and v such that every vertex of $V - \{u, v\}$ lies in some $u - v$ detour

Proof. Every vertex of $V - \{u, v\}$ lies in some $u - v$ detour implies $S = \{u, v\}$ is a detour set of G. Also, $diam_D < 4$ implies S dominates all the vertices of $V - S$. Hence, S is a detour dominating set of G so that $\gamma_{dn}(G) \leq 2$. But, always, $\gamma_{dn}(G) \geq dn(G) \geq 2$. Therefore, $\gamma_{dn}(G) = 2$.

Conversely, Suppose $\gamma_{dn}(G) = 2$. Let $S = \{u, v\}$ be a minimum detour set of G. Now, S is detour implies every vertex of $V - S$ must lies in some $u - v$ detour and further S is a dominating set implies every vertex of $V - S$ is dominated by S. Hence, u and v are antipodal vertices of G and so $d(u, v) = diameter_D$ of G.

If $diam_D \geq 4$, every detour P joining u and v must be of length greater than or equal to 4 and so there exists atleast one vertex in $V - S$ which is not dominated by S. This is a contradiction to the fact that S is a γ_{dn} – set of G. Hence the result.

VI. ON DETOUR DOMINATION NUMBER OF EDGE ADDED GRAPHS

Here, it is studied that how the detour domination number of a non-complete connected graph is affected by addition of a single vertex with one edge

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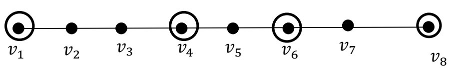
incident with this vertex. Throughout this section, $G \circ K_2$ represents the graph obtained from G by adjoining an edge with some vertex of G.

Theorem 6.1: Let G be any non-complete connected graphs. Let $G' = G \circ K_2$ be a graph obtained from G by adjoining an edge with some vertex of G. Then, $\gamma_{dn}(G') \leq \gamma_{dn}(G) + 1$.

Proof: Let $G' = G \circ K_2$. Let $V(G') = V(G) \cup \{u\}$ and $E(G') = E(G) \cup \{uv\}$ for some $v \in V(G)$. If S is any minimum detour dominating set of G, then $S \cup \{u\}$ is a detour dominating set of G'. Therefore, $\gamma_{dn}(G') \leq |S \cup \{u\}| = |S| + 1 = \gamma_{dn}(G) + 1$.

Remark 6.2: In the above theorem, both the upper and lower bounds for $\gamma_{dn}(G')$ are sharp.

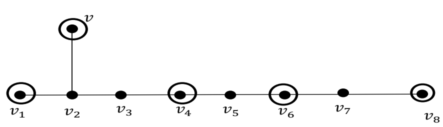
For example, consider $G = P_8, G, G'$ *and* G'' are as in figures 6.2(a), 6.2(b) and 6.2(c) respectively.



G Figure 6.2 (a)

$\{v_1, v_4, v_6, v_8\}$is a detour dominating set of P_8.

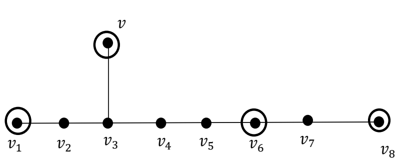
Therefore,$\gamma_{dn}(P_8) = \left\lceil \frac{8-4}{3} \right\rceil + 2 = 2 + 2 = 4$



G' Figure 6.2 (b)

$\{v, v_1, v_4, v_6, v_8\}$is a detour dominating set ofG'.

Therefore,$\gamma_{dn}(G') = 5 = \gamma_{dn}(P_8) + 1$.



G'' Figure 6.2 (c)

$\{v, v_1, v_5, v_8\}$is a detour dominating set of G''.

Therefore,$\gamma_{dn}(G'') = 4 = \gamma_{dn}(P_8)$.

Theorem 6.3: If a vertex is joined by an edge to any vertex ofP_n, where $n = 3k + 1$ *and* $k \geq 1$, then for the resulting graph $G' = P_n \circ K_2, \gamma_{dn}(G') = \gamma_{dn}(P_n) + 1$.

Proof:

Case 1: suppose G' is the graph obtained form P_n by adding an edge to one of the end vertices of P_n.

In this case, $G' \cong P_{n+1.}$ Therefore,

$\gamma_{dn}(G') = \gamma_{dn}(P_{n+1.}) = \left\lceil \frac{(n+1)-4}{3} \right\rceil + 2 = \left\lceil \frac{3k+2-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-2}{3} \right\rceil + 2 = k + 2$and

$$\gamma_{dn}(G) = \gamma_{dn}(P_{n.}) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-3}{3} \right\rceil + 2 = k - 1 + 2 = k + 1.$$

Hence, $\gamma_{dn}(G') = \gamma_{dn}(G) + 1$.

Case 2: Suppose G' is obtained by adding an edge to one of the internal of P_n.

In this case, the number of end vertices of G' is 3. Therefore, every minimum detour dominating set of G' contains these three end vertices. Clearly, any minimum dominating set of a path of $n-4$ or $n-5$ vertices along with these three end vertices forms a minimum detour dominating set of G' and so $\gamma_{dn}(G') = 3 + \left\lceil \frac{n-4}{3} \right\rceil = 3 + \left\lceil \frac{3k+1-4}{3} \right\rceil = 3 + k - 1 = k + 2 = \gamma_{dn}(G) + 1$, as $\left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{n-5}{3} \right\rceil$ when $n = 3k + 1$.

Theorem 6.4: Let G =P_n, $n > 3$, and let $n = 3k$. If G' is obtained by adding an edge to one of the end vertices of G, then $\gamma_{dn}(G') = \gamma_{dn}(G)$ and if an edge is added to one of its internal vertices, then $\gamma_{dn}(G') = \gamma_{dn}(G) + 1$.

Proof:

Case 1: Suppose G' is obtained by adding an edge to one of the end vertices of $G = P_n$.

In this case, $G' \cong P_{n+1.}$ Therefore,

$\gamma_{dn}(G') = \gamma_{dn}(P_{n+1.}) = \left\lceil \frac{(n+1)-4}{3} \right\rceil + 2 = \left\lceil \frac{3k+1-4}{3} \right\rceil + 2 = k - 1 + 2 = k + 1$and

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$$\gamma_{dn}(G) = \gamma_{dn}(P_{n.}) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{3k-4}{3} \right\rceil + 2$$
$$= \left\lceil \frac{3(k-1)-1}{3} \right\rceil + 2$$
$$= k - 1 + 2 \qquad = k + 1.$$

Hence, $\gamma_{dn}(G^{'}) = \gamma_{dn}(G)$.

Case 2: Suppose $G^{'}$ is obtained by adding an edge to any one of the internal vertices of $G = P_n$.

As in theorem 6.4.Clearly, any minimum dominating set of a path of $n-4$ or $n-5$ vertices along with these three end vertices forms a minimumdetour dominating set of $G^{'}$ Therefore, $\gamma_{dn}(G^{'}) = 3 + \left\lceil \frac{n-4}{3} \right\rceil = 3 + \left\lceil \frac{3k-4}{3} \right\rceil = 3 + \left\lceil \frac{3(k-1)-4}{3} \right\rceil = 3 + k - 1 = k + 2 = \gamma_{dn}(G) + 1$, as $\left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{n-5}{3} \right\rceil$ when $n = 3k$.

Theorem 6.5: For a complete graph K_p, $\gamma_{dn}(K_p - e=2$ for every edge e in Kp.

Proof. Let $e = uv \in E(K_p)$. Suppose$S = \{u, v\}$, then for every vertex $w \in V(K_p - e) - S$, there exists a $u - v$ detour of length 2 containing w. As $\deg(u) = \deg(v) = p - 2$ in $K_p - e$, S is a detour dominating set of $K_p - e$ and so $\gamma_{dn}(K_p - \{e\}) \leq 2$. Hence, by Remark 2.5,$\gamma_{dn}(K_p - \{e\}) = 2$

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Author Profile

Dr. A. NellaiMurugan, Associate Professor, S.S.Pillai Centre for Research in Mathematics, Department of Mathematics, V.O.Chidambaram College, Thoothukudi-628 008 Tamilnadu, India, College in affiliated to Manonmaniumsundaranar University, Tirunelveli-12,TamilNadu. He has thirty two years of Post Graduate teaching experience in twelve years of Research experience. He is guiding Six PhD Scholars. He has Published more than Seventy research papers in reputed national and international journals.

A.Esakkimuthu, Full Time Research Scholar, Dr.S.S.Pillai Centre for Research in Mathematics, V. O. Chidambaram college, Thoothukudi-628 008, Tamilnadu, India , He has published three research articles is reputed National and International Journals and communicated twelve more articles to various reputed Journals.

Dr.G.Mahadevan, M.Sc.,M.Phil.,M.Tech.,Ph.D., is having 21 years of teaching experience in various colleges and Universities including Head of the Dept. of Mathematics at Anna University, Tirunelveli, Tirunelveli Region, Tirunelveli. Currently he is working as Asst.Professor in the dept of Mathematics Gandhigram Rural Institute-Deemed University,(Fully funded by Minsitry of Human Resource Development, Government of India, Gandhigram. He has Published more than 65 Research Papers in Various International/National Journals. Three scholars have been awarded their Ph.D., under his guidance and many of them are doing their Ph.D., degree under his guidance. Recently he received Best Faculty Award-Senior Category in Mathematics by former UGC Vice Chairman at Nehru Garden, during 2013.He has also receivedDr.A.P..J Abdul Kalam Award for Scientific Excellence-2015.

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