ON GR*-CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT
In this paper, we introduce the concept of fuzzy generalized regular star closed sets and study some of their properties.

Keywords: Fuzzy rg-closed sets, Fuzzy gr*-closed sets, Fuzzy gr*-open sets.

AMS Suject Classification: 94D05, 54C10, 54C08, 54C05

1. INTRODUCTION

Zadeh [10] introduce the concept of fuzzy sets. The study of fuzzy topology has introduced by Chang[1]. In 1995, Thakur S.S [8] introduced the concept of fuzzy generalized closed sets. Later on Jin Han Park[3] studied the concept of regular generalized closed in fuzzy topological space. In this paper, the concept of fuzzy generalized regular star closed sets is introduced and studied. Some properties are proved and their relations with different fuzzy sets in fuzzy topological spaces are investigated.

2 PRELIMINARIES

Definition 2.1: A fuzzy subset λ of a fuzzy topological space (X, τ) is called

1. A fuzzy generalized closed set [8] if $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open in (X, τ).
2. A fuzzy generalized semi-closed set [4] if $scl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open in (X, τ).
3. A fuzzy semi generalized closed set [5] if $scl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy semi-open in (X, τ).
4. A fuzzy generalized semi-pre closed set [6] if $spcl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open in (X, τ).
5. A fuzzy generalized pre closed set if [2] $pcl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open in (X, τ).
6. A fuzzy regular generalized closed set [3] if $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy regular open in (X, τ).
7. A fuzzy generalized pre regular closed set [9] if $pcl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy regular open in (X, τ).
8. A fuzzy αgeneralized closed set [7] if $\alpha cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open in (X, τ).

3. FUZZY gr*-CLOSED SETS

Definition 3.1: A fuzzy set λ of a fuzzy topological spaces (X, τ) is called a fuzzy gr* -closed set if $Rcl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy g-open subset of X.

Example 3.2: Consider X = {a, b} and $\tau = \{1, 0, \lambda, \nu, \alpha\}$ is a fuzzy topology on X. The fuzzy sets are defined as $\lambda = \frac{0.6}{a} + \frac{0.5}{b}$,

ISSN: 2278 – 7798

$\nu = \frac{0.8}{a} + \frac{0.6}{b}$, $\alpha = \frac{0.2}{a} + \frac{0.3}{b}$. Here λ and ν are fuzzy gr* -closed sets but α is not fuzzy gr* -closed set.

Theorem 3.3: Every fuzzy closed set in X is fuzzy gr* -closed set in X.

Proof: Let λ be a fuzzy closed set in X. Let μ be a fuzzy g-open such that λ ≤ μ. Since λ is fuzzy closed, that is cl(λ) = λ, cl(λ) ≤ μ. But cl(λ) ≤ Rcl(λ) ≤ μ. Therefore Rcl(λ) ≤ μ. Hence λ is fuzzy gr* -closed set in X.

Theorem 3.4: Every fuzzy r-closed set in X is fuzzy gr* -closed in X.

Proof: Let λ be fuzzy r-closed set in X. Let μ be fuzzy g-open such that λ ≤ μ. Since λ is fuzzy r-closed, we have Rcl(λ) = λ ≤ μ. Therefore Rcl(λ) ≤ μ. Hence λ is fuzzy gr* -closed set in X.

Remark 3.5: The converse of the above theorem 3.3 and 3.4 need not be true as seen in the following example.

Example 3.6: Consider X = {a, b} and τ = {1, 0, λ, ν, α} is a fuzzy topology on X. The fuzzy sets are defined as $\lambda = \frac{0.6}{a} + \frac{0.5}{b}$, $\nu = \frac{0.8}{a} + \frac{0.6}{b}$, $\alpha = \frac{0.2}{a} + \frac{0.3}{b}$. Here λ and ν are fuzzy gr*-closed sets but not fuzzy closed and fuzzy r-closed sets.

Theorem 3.7: For a fuzzy topological spaces (X, τ) the following conditions are hold

(i) Every fuzzy gr*-closed set is fuzzy gs-closed set.
(ii) Every fuzzy gr*-closed set is fuzzy sg-closed set.
(iii) Every fuzzy gr*-closed set is fuzzy gp-closed set.
(iv) Every fuzzy gr*-closed set is fuzzy gsp-closed set.
(v) Every fuzzy gr*-closed set is fuzzy rg-closed set.
(vi) Every fuzzy gr*-closed set is fuzzy gpr-closed set.
(vii) Every fuzzy gr*-closed set is fuzzy αg-closed set.
(viii) Every fuzzy gr*-closed set is fuzzy g-closed set.

Proof. (i) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy open set such that λ ≤ μ. Since every fuzzy open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have scl(λ) ≤ rcl(λ) ≤ μ. Therefore scl(λ) ≤ μ. Hence λ is fuzzy gs-closed set in X.

(ii) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy semi-open set such that λ ≤ μ. Since every fuzzy semi-open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have scl(λ) ≤ rcl(λ) ≤ μ. Therefore scl(λ) ≤ μ. Hence λ is fuzzy sg-closed set in X.

(iii) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy open set such that λ ≤ μ. Since every fuzzy open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have pcl(λ) ≤ rcl(λ) ≤ μ. Therefore pcl(λ) ≤ μ. Hence λ is fuzzy gp-closed set in X.

(iv) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy open set such that λ ≤ μ. Since every fuzzy regular open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have spcl(λ) ≤ rcl(λ) ≤ μ. Therefore spcl(λ) ≤ μ. Hence λ is fuzzy gsp-closed set in X.

(v) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy regular open set such that λ ≤ μ. Since every fuzzy open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have cl(λ) ≤ rcl(λ) ≤ μ. Therefore cl(λ) ≤ μ. Hence λ is fuzzy rg-closed set in X.

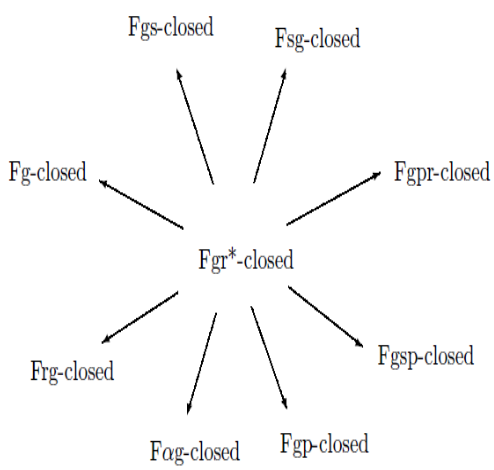
(vi) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy regular open set such that λ ≤ μ. Since every fuzzy regular open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have pcl(λ) ≤ rcl(λ) ≤ μ. Therefore pcl(λ) ≤ μ. Hence λ is fuzzy gpr-closed set in X.

(vii) Let λ be fuzzy gr*-closed set in X. Let μ be fuzzy open set such that λ ≤ μ. Since

ISSN: 2278 – 7798

every fuzzy open set is fuzzy g-open and λ is fuzzy gr*-closed set. We have αcl(λ) ≤ rcl(λ) ≤ μ. Therefore αcl(λ) ≤ μ. Hence λ is fuzzy αg-closed set in X.
(viii) It is obvious.

For the above theorem, we have the following implications hold.



Remark 3.8: In the above implication diagram the reverse implications need not be true as seen in the following examples.

Example 3.9: Consider X = {a, b} and τ = {1, 0, ν} is a fuzzy topology on (X, τ). The fuzzy sets are defined as $\lambda = \frac{0.2}{a} + \frac{0.9}{b}$, $\nu = \frac{0.5}{a} + \frac{0.5}{b}$. Here λ is fg-closed, fsg-closed, fgs-closed, fαg-closed, fgp-closed, fgsp-closed, fgpr-closed, frg-closed but not fuzzy gr*-closed set in (X, τ).

Example 3.10: Consider X = {a, b} and τ = {1, 0, λ} is a fuzzy topology on (X, τ). The fuzzy sets are defined as $\lambda = \frac{0.6}{a} + \frac{0.5}{b}$, $\nu = \frac{0.7}{a} + \frac{0.8}{b}$, $\alpha = \frac{0.3}{a} + \frac{0.4}{b}$. Here α is fg-closed but not fuzzy gr*-closed set in (X, τ).

Theorem 3.11: Let λ be a fuzzy g-open subset of (X, τ), Then λ is fuzzy r-closed set if λ is fuzzy gr*-closed set.
Proof: Let λ be a fuzzy g-open subset of (X, τ). Let λ be fuzzy gr*-closed set. Then by definition of fuzzy gr*-closed set, if Rcl(λ) ≤ μ whenever λ ≤ μ and μ is fuzzy g-open subset of (X, τ). Since λ is fuzzy g-open, Rcl(λ) = λ. Therefore λ is fuzzy r-closed set.

Theorem 3.12: A fuzzy subset λ of (X, τ) is a fuzzy gr*-closed set if it is a fuzzy r-closed set.
Proof: Let λ be a fuzzy r-closed set. Let μ be a fuzzy g-open subset such that λ ≤ μ. Since λ is fuzzy r-closed, we have Rcl(λ) = λ ≤ μ. Therefore Rcl(λ) ≤ μ. Hence λ is fuzzy gr*-closed set.

Remark 3.13: The converse of the above theorem need not be true as seen in the following example.

Example 3.14: Consider X = {a, b} and τ = {1, 0, λ, ν, α} is a fuzzy topology on X. The fuzzy sets are defined as $\lambda = \frac{0.6}{a} + \frac{0.5}{b}$, $\nu = \frac{0.8}{a} + \frac{0.6}{b}$, $\alpha = \frac{0.2}{a} + \frac{0.3}{b}$. Here λ and ν are fuzzy gr*-closed sets but not fuzzy r-closed set.

Theorem 3.15: The finite union of the fuzzy gr*-closed sets is fuzzy gr*-closed.
Proof: Let λ and μ be fuzzy gr*-closed sets in X. Let μ be a fuzzy g-open in X such that λ∪ν ≤ μ. Then λ ≤ μ and ν ≤ μ. Since λ and ν are fuzzy gr*-closed set, Rcl(λ) ≤ μ and Rcl(ν) ≤ μ. We have Rcl(λ ∪ ν) = Rcl(λ) ∪ Rcl(ν) ≤ μ. Therefore Rcl(λ ∪ ν) ≤ μ. Hence λ ∪ ν is fuzzy gr*-closed set in X.

ISSN: 2278 – 7798

Theorem 3.16: The finite intersection of two fuzzy gr*-closed sets is fuzzy gr*-closed.
Proof: The proof is obvious.

Theorem 3.17: The intersection of a fuzzy gr*-closed set and a fuzzy closed set is a fuzzy generalized closed set.
Proof: Let λ be a fuzzy gr*-closed subset of X. Let α is a fuzzy closed set. If μ is an fuzzy g-open subset of X with λ ∪ α ≤ μ, then λ ≤ μ ∪ (X\α). So, Rcl(λ) ≤ μ ∪ (X\α). Then cl(λ ∪ α) = cl(λ) ∩ cl(α) ≤ Rcl(λ) ∩ cl(α) = Rcl(λ) ∩ α ≤ μ. So, λ ∩ α is a fuzzy generalized closed set in X.

Remark 3.18: The intersection of a fuzzy gr*-closed set and a fuzzy regular closed set is a fuzzy gr*-closed set.

Theorem 3.19: Let λ ≤ ν ≤ Rcl(λ) and λ is a fuzzy gr*-closed subset of (X, τ), then ν is also a fuzzy gr*-closed subset of (X, τ).
Proof: Since λ is a fuzzy gr*-closed subset of (X, τ), we have Rcl(λ) ≤ μ, whenever λ ≤ μ and μ is an fuzzy g-open subset of X. Let λ ≤ ν ≤ Rcl(λ). i.e) Rcl(λ) = Rcl(ν).
Let if possible, there exists a fuzzy open subset η of X such that ν ≤ η. So, λ ≤ μ and ν is fuzzy gr*-closed subset of X. Rcl(λ) ≤ η i.e) Rcl(ν) ≤ η. Hence ν is also a fuzzy gr*-closed subset of X.

Theorem 3.20: Let λ ≤ ν ≤ X, where ν is fuzzy g-open and fuzzy gr*-closed set in X. If λ is fuzzy gr*-closed in ν then λ is fuzzy gr*-closed in X.
Proof: Let μ be a fuzzy g-open set of X such that λ ≤ μ. Since λ ≤ μ∩ν where μ∩ν is fuzzy gr*-closed in ν, Rcl(λ) ≤ μ∩ν holds. We have Rcl(λ)∩ν ≤ μ∩ν. Since λ ≤ ν, we have Rcl(λ) ≤ Rcl(ν). Since ν is fuzzy g-open and fuzzy gr*-closed in X, by Theorem 3.11 , we have ν is fuzzy r-closed. Therefore Rcl(ν) = ν. Thus Rcl(λ) ≤ ν implies Rcl(λ) = Rcl(λ) ∩ ν ≤ μ ∩ ν ≤ μ. Hence λ is fuzzy gr*-closed in X.

Theorem 3.21: A fuzzy subset λ of X is fuzzy gr*-closed set in X iff Rcl(λ)−λ contains no non-empty fuzzy g-closed set in X.
Proof: Suppose that F is a non-empty fuzzy g-closed subset if Rcl(λ)−λ. Now F ≤ Rcl(λ) ∩ λ^c. Therefore F ≤ λ and F ≤ λ^c. Since F^c is fuzzy g-open set and λ is fuzzy gr*-closed. Rcl(λ) ≤ F^c i.e) F ≤ $Rcl(\lambda)^c$. Hence F ≤ Rcl(λ) ∩ $[Rcl(\lambda)]^c$ = φ. i.e) F = φ. Thus Rcl(λ) − λ contains no non-empty fuzzy g-closed set.
Conversely, assume that Rcl(λ) − λ contains no non-empty fuzzy g-closed set. Let λ ≤ μ, μ is fuzzy g-open. Suppose that Rcl(λ) is not contained in μ. Then Rcl(λ) ∩ μ^c is a non-empty fuzzy g-closed set and contained in Rcl(λ) − λ. Therefore Rcl(λ) ≤ μ. Hence λ is fuzzy gr*-closed set.

4. FUZZY gr*-OPEN SETS

Definition 4.1: A fuzzy subset λ of a topological space X is called fuzzy gr*-open set if λ^c is fuzzy gr*-open set in X.

Example 4.2: Consider X = {a, b} and τ = {1, 0, λ, ν, α} is a fuzzy topology on X. The fuzzy sets are defined as λ = $\frac{0.6}{a}+\frac{0.5}{b}$, ν = $\frac{0.8}{a}+\frac{0.6}{b}$, α = $\frac{0.2}{a}+\frac{0.3}{b}$. Here λ^c and ν^c are fuzzy gr* -open sets but α^c is not fuzzy gr* -open set.

Theorem 4.3. A fuzzy subset λ^c of a fuzzy topological space (X, τ) is fuzzy
gr*-open iff μ ≤ $Rint(\lambda)^c$ where μ is fuzzy g-closed in X and μ ≤ λ^c.
Proof: **Necessity:** Suppose μ ≤ $Rint(\lambda)^c$ where μ is fuzzy g-closed in (X, τ) and μ ≤

ISSN: 2278 – 7798

λ^c. Let $(\lambda^c)^c \leq$ M, where M is fuzzy g-open. Hence $M^c \leq \lambda^c$, where M^c is fuzzy g-closed. Hence by assumption $M^c \leq$ Rint(λ^c), which implies $[\text{Rint}(\lambda^c)]^c \leq$ M. Therefore Rcl(λ) ≤ M. Thus λ is fuzzy gr*-closed implies λ^c is fuzzy gr*-open set.

Sufficiency: Let λ^c is fuzzy gr*-open in X with $N \leq \lambda^c$, where N is fuzzy g-closed. We have λ is fuzzy gr*-closed with $\lambda \leq N^c$ where N^c is fuzzy g-open. Then we have Rcl(λ) ≤ N^c implies N ≤ X−Rcl(λ) = Rint(X−λ) = Rint(λ^c). Hence proved.

Theorem 4.4: Every fuzzy r-open set is fuzzy gr*-open set.

Proof: Let λ^c be a fuzzy r-open set. Then X − λ^c is fuzzy r-closed. by Theorem 3.4, X −λ^c is fuzzy gr*-closed. Hence λ^c is fuzzy gr*-open set.

Theorem 4.5: If Rint(λ^c) ≤ $\nu^c \leq \lambda^c$ and λ^c is a fuzzy gr*-open subset of (X, τ), then ν^c is also a fuzzy gr*open set of (X, τ).

Proof: Rint(λ^c) ≤ $\nu^c \leq \lambda^c$ implies λ ≤ ν ≤ Rcl(λ). Given is fuzzy gr*-closed. by Theorem 3.19 , ν is fuzzy gr*-closed. Therefore νc is fuzzy gr*-open set.

Theorem 4.6: If a subset λ of a fuzzy topological space (X, τ) is fuzzy gr*-closed, then Rcl(λ) − λ is fuzzy gr*-open.

Proof: Let λ ≤ X be a fuzzy gr*-closed and F be fuzzy g-closed set such that F ≤ Rcl(λ)−λ. Then by Theorem 3.21 , F = φ. So φ = F ≤ Rint(Rcl(λ)−λ). Thus λ is fuzzy gr*-open set.

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ISSN: 2278 – 7798