Negative Jacobsthal Numbers

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***Abstract*—Here We express Jacobsthal and Jacobsthal Lucas Numbers using Binet formula and using this formula we discuss some Properties.**

***IndexTerms*—Jacobsthal Numbers, Negative Jacobsthal Numbers,Sum of Squares.**

1. INTRODUCTION

The Jacobsthal and Jacobsthal –Lucas sequences J_n and j_n are defined by the recurrence relations

$$J_0 = 0, J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2.$$

---------------(1)

$$j_0 = 2, j_1 = 1,\ j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2.$$

---------------(2)

Applications of these two sequences to curves are given in [1]. Sequence (1) appears in [2] but (2) does not. From (1) and (2) we thus have the following tabulation for the Jacobsthal numbers J_n and the Jacobsthal –Lucas sequences j_n.

n	0	1	2	3	4	5	6	7	8	9	10 …
J_n	0	1	1	3	5	11	21	43	85	171	341 …
j_n	2	1	5	7	17	31	65	127	257	511	1025…

When required we can extend these sequences through negative values of n by means of the recurrence (1) and (2). Observe that all the J_n and j_n except j_0 are odd by virtue of the definitions.

In [3], the Binet forms of Jacobsthal form are given as

Let n be an integer.The Binet-like formulas of the negative Jacobsthal and negative Jacobsthal Lucas numbers are

$$J_{-n} = \frac{2^{-n} - (-1)^{-n}}{3}$$

And $j_{-n} = 2^{-n} + (-1)^{-n}$

n	0	1	2	3	4	5	6	7	8	9	10…
J_{-n}	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{11}{32}$	$-\frac{21}{64}$	$\frac{43}{128}$	$-\frac{85}{256}$	$\frac{171}{512}$	$-\frac{341}{1024}$ …
j_{-n}	2	$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{7}{8}$	$\frac{17}{16}$	$-\frac{31}{32}$	$\frac{65}{64}$	$-\frac{127}{128}$	$\frac{257}{256}$	$-\frac{511}{512}$	$\frac{1025}{1024}$ …

Negative Jacobsthal Numbers and Jacobsthal Lucas Numbers can also defined as follows

$$J_{-(2n+1)} = \frac{3p_1 + (2^{2n} - 1)q_1}{3.\,2^{2n} q_1}$$

$$J_{-2n} = -\left[\frac{3p_1 + (2^{2n} - 1)q_1}{3.\,2^{2n} q_1}\right]$$

with the initial conditions $J_{-1} = \frac{p_1}{q_1} = \frac{1}{2}$ and $J_{-2} = \frac{p_2}{q_2} = -\frac{1}{4}$

$$j_{-n} = \frac{2^n + 1}{2^n}$$

$$j_{-n} = (-1)^n \left(\frac{2^n + 1}{2^n}\right)$$

The recurrence relation satisfied the Jacobsthal Numbers is

$J_{-(n-1)} - J_{-n} = 2J_{-(n+1)}$, $J_{-1} = \frac{1}{2}$ and $J_{-2} = -\frac{1}{4}$

$j_{-(n-1)} - j_{-n} = 2j_{-(n+1)}$, $j_{-1} = -\frac{1}{2}$ and $j_{-2} = \frac{5}{4}$

Proposition 1.1

$$J_{-n}^2 + J_{-(n+1)}^2 = \frac{1}{9}\{2[j_{-2n-2}] + 2^{-2n-2}.3 - 2^{-n}(-1)^{-n}\}$$

Proof

Let $J_{-n} = \frac{2^{-n} - (-1)^{-n}}{3}$

$$J_{-n}^2 + J_{-(n+1)}^2 = \left[\frac{2^{-n} - (-1)^{-n}}{3}\right]^2 + \left[\frac{2^{-(n+1)} - (-1)^{-(n+1)}}{3}\right]^2$$

$$= \frac{1}{9}\{[2^{-2n}(1 + 2^{-2})] + [(-1)^{-2n}(1 + (-1)^{-2})] - 2[2^{-n}(-1)^{-n}[1 + 2^{-1}(-1)^{-1}]]\}$$

$$= \frac{1}{9}\left\{2^{-2n}\left(\frac{5}{4}\right) + (-1)^{-2n}.2 - 2(2)^{-n}(-1)^{-n}\left(\frac{1}{2}\right)\right\}$$

$$= \frac{1}{9}\{2^{-2n-2}.5 + 2(-1)^{-2n-2} - 2^{-n}(-1)^{-n}\}$$

$$= \frac{1}{9}\{2^{-2n-2}.2 + 2(-1)^{-2n-2} + 2^{-2n-2}.3 - 2^{-n}(-1)^{-n}\}$$

$$= \frac{1}{9}\{2[2^{-2n-2} + (-1)^{-2n-2}] + 2^{-2n-2}.3 - 2^{-n}(-1)^{-n}\}$$

$$J_{-n}^2 + J_{-(n+1)}^2 = \frac{1}{9}\{2[j_{-2n-2}] + 2^{-2n-2}.3 - 2^{-n}(-1)^{-n}\}$$

Proposition 1.2

$$j_{-n}^2 + j_{-(n+1)}^2 = 2(j_{-2n-2}) + 2^{-n}(-1)^{-n} + 3.\,2^{-2n-2}$$

Proof

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Let $j_{-n} = 2^{-n} + (-1)^{-n}$

$$= [2^{-n} + (-1)^{-n}]^2 + [2^{-(n+1)} + (-1)^{-(n+1)}]^2$$

$$= 2^{-2n}[1+2^{-2}] + (-1)^{-2n}[1+2^{-2}] + 2.2^{-n}.(-1)^{-n}[1-2^{-1}]$$

$$= 2^{-2n}\left[\frac{5}{4}\right] + 2(-1)^{-2n} + 2^{-n}.(-1)^{-n}$$

$$= 2[2^{-2n-2} + (-1)^{-2n-2}] + 3(2^{-2n-2}) + 2^{-n}.(-1)^{-n}$$

$$= 2[j_{-2n-2}] + 3(2^{-2n-2}) + 2^{-n}.(-1)^{-n}$$

Proposition1.3

$$J_{-(n+1)}J_{-(n-1)} - J_{-n}^2 = 2^{-(n+1)}(-1)^{-n}$$

Proof

$$= \left(\frac{2^{-(n+1)} - (-1)^{-(n+1)}}{3}\right)\left(\frac{2^{-(n-1)} - (-1)^{-(n-1)}}{3}\right) - \left(\frac{2^{-n} - (-1)^{-n}}{3}\right)^2$$

$$= \frac{1}{9}\{[2^{-2n} + (-1)^{-2n}] - [2^{-n}(-1)^{-n}(2^{-1}(-1) + 2(-1)^{-1})]\}$$

$$-\frac{1}{9}\{2^{-2n} + (-1)^{-2n} - 2.2^{-n}(-1)^{-n}\}$$

$$= \frac{1}{9}\left\{2.2^{-n}(-1)^{-n} - 2^{-n}(-1)^{-n}\left(-\frac{1}{2} - 2\right)\right\}$$

$$= \frac{1}{9}\left\{2^{-n}(-1)^{-n}\left(2 + \frac{1}{2} + 2\right)\right\}$$

$$= \frac{1}{9}\{9.2^{-n-1}(-1)^{-n}\}$$

$$J_{-(n+1)}J_{-(n-1)} - J_{-n}^2 = 2^{-(n+1)}(-1)^{-n}$$

Proposition1.4

$$J_{-n} + J_{-(n+1)} = 2^{-(n+1)}$$

Proof

$$J_{-n} + J_{-(n+1)} = \left(\frac{2^{-n} - (-1)^{-n}}{3}\right) + \left(\frac{2^{-(n+1)} - (-1)^{-(n+1)}}{3}\right)$$

$$= \frac{1}{3}\left[2^{-n}\left(1 + \frac{1}{2}\right) - (-1)^{-n} + (-1)^{-n}\right]$$

$$J_{-n} + J_{-(n+1)} = 2^{-(n+1)}$$

Remark:

$$j_{-n} + j_{-(n+1)} = 3.2^{-(n+1)}$$

Proposition1.5

$$J_{-(n+1)} - J_{-n} = -2J_{-(n+2)}$$

Proof

$$J_{-(n+1)} - J_{-n} = \left(\frac{2^{-(n+1)} - (-1)^{-(n+1)}}{3}\right) - \left(\frac{2^{-n} - (-1)^{-n}}{3}\right)$$

$$= \frac{1}{3}\left[2^{-n}\left(\frac{1}{2} - 1\right) - (-1)^{-n}(-2)\right]$$

$$= \frac{1}{3}\left[2^{-n}\left(-\frac{1}{2}\right) - (-1)^{-n}(-2)\right]$$

$$= \frac{2}{3}\left[-2^{-(n+2)} + (-1)^{-(n+2)}\right]$$

$$= -2J_{-(n+2)}$$

Remark

$$j_{-(n+1)} - j_{-n} = -2j_{-(n+2)}$$

Proposition1.6

$$J_{-m}j_{-n} - J_{-n}j_{-m} = 0$$

Proof

$$J_{-m}j_{-n} - J_{-n}j_{-m} = \left(\frac{2^{-m} - (-1)^{-m}}{3}\right)(2^{-n} + (-1)^{-n}) - \left(\frac{2^{-n} - (-1)^{-n}}{3}\right)(2^{-m} + (-1)^{-m})$$

$$= \frac{1}{3}\{2^{-m}2^{-n} - 2^{-m}2^{-n}\}$$

$$J_{-m}j_{-n} - J_{-n}j_{-m} = 0$$

Proposition1.7

$$J_{-m}j_{-n} + J_{-n}j_{-m} = 2J_{-m}j_{-n}$$

Proof

$$= \left(\frac{2^{-m} - (-1)^{-m}}{3}\right)(2^{-n} + (-1)^{-n}) - \left(\frac{2^{-n} - (-1)^{-n}}{3}\right)(2^{-m} + (-1)^{-m})$$

$$= \frac{1}{3}\{2^{-m}2^{-n} + 2^{-m}(-1)^{-n} - 2^{-n}(-1)^{-m} - (-1)^{-m}(-1)^{-n} + 2^{-m}2^{-n} + 2^{-m}(-1)^{-n}$$

$$-2^{-n}(-1)^{-m} - (-1)^{-m}(-1)^{-n}\}$$

$$= \frac{2}{3}\{2^{-m}2^{-n} + 2^{-m}(-1)^{-n} - 2^{-n}(-1)^{-m} - (-1)^{-m}(-1)^{-n}\}$$

$$= \frac{2}{3}\{2^{-m}(2^{-n} + (-1)^{-n}) - (-1)^{-m}(2^{-n} + (-1)^{-n})\}$$

$$= \frac{2}{3}\{2^{-m}j_{-n} - (-1)^{-m}j_{-n}\}$$

$$= \frac{2}{3}\{j_{-n}(2^{-m} - (-1)^{-m})\}$$

$$= \frac{2}{3}\{j_{-n}.3J_{-m}\}$$

$$J_{-m}j_{-n} + J_{-n}j_{-m} = 2\{j_{-n}J_{-m}\}$$

REFERENCES

[1] A.F.Horadam, "Jacobsthal and pell curves" The Quartely 26.I(1988):79-83.
[2] N.J.A.Sloane, A Handbook of integer Sequences.NewYork:Academic Press,1973.
[3] A.F.Horadam, Jacobsthal Representation Numbers,Unversity of New England,Armidale,2351,Australia.
[4] Zvonko Cerin, Sums of Squares and Products of Jacobsthal Numbers,Journal of integer Sequences,Vol. 10 (2007)
[5] F. Koken, D. Bozkurt, "On the Jacobsthal numbers by matrix methods", Int. J. Contemp. Math. Sciences, 2008; 30(3), 605-614.
[6] T. Koshy, "Fibonacci and Lucas number with application", wiley, New York, 2001.

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