ISSN: 2278 – 7798
International Journal of Science, Engineering and Technology Research (IJSETR)
Volume 5, Issue 4, April 2016

$K_{3,3q}$ graph - Edge Product Number

J.P. Thavamani
Associate Professor, Department of Mathematics M.E.S. College, Nedumkandam(po)
Idukki(dt), Kerala(st)-685553, India

ABSTRACT: ***The edge product number of G is the minimum number of K_2 components that added to G, the resulting graph is an edge product graph, it is represented by EPN(G). An edge function f: E $\to$ P and its corresponding edge product function F which make $G \cup rK_2$ an edge product graph are called an optimal edge function and an optimal edge product function of G respectively. In this paper we establish that the graph $K_{3,3q}$ is an edge product graph and the edge product number of $K_{3,3q}$ is found if q is even or odd.***

KEYWORDS: bipartite graph, edge function, edge product function, edge product graph

I. INTRODUCTION

A graph labelling is an injective mapping from elements of a graph (can be vertices, edges, faces, or combination) to a set of numbers (usually positive integers). Labelled graph serve as useful models for a broad range of applications such as coding theory, X- ray, radar, astronomy, crystallography, circuit design, communication network and data base management. Labelling of graph is an assignment of integers either to the vertices of G or edges of G or both subject to certain condition. A lot of research has been done on various types of labelling by applying to various graphs and still there are many open problems in this field. By a graph G = (V, E) we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Harary [3] and basic number theoretic results we refer to [1]. Magic labelling is surveyed and edge magic total labelling was first discussed by Kotzig and Rosa [2]. For recent survey on graph labelling we refer to Gallian [4]. We investigated the edge product number of complete graphs in [5]. In this paper, we determine the edge product number of $K_{3,3q}$ graph for all positive integer q.

II. RELATED WORK

Definitions: 2.1

Let G be a graph and P be a set of positive integers with |E| = |P|. Then any bijection f: E $\to$ P is called an edge function of the graph G.

If $F(v)=\prod\{f(e)/e$ is incident on v$\}$on V then F is called the edge product function of the edge function f. Let G be a graph and G is said to be an edge product graph if there exists an edge function f: E $\to$ P such that f and its corresponding edge product function F on V satisfy the conditions that $F(v)\in P$ for every $v\in V$ and if $e_1,e_2, \ldots,e_p\in E$ such that $f(e_1).f(e_2). \ldots .f(e_p)\in P$, then $e_1, e_2, \ldots,e_p$ are incident on a vertex.

The edge product number of G, denote the minimum number r of K_2 components that must be added to G so that the resulting graph is an edge product graph, it is represented by EPN(G). Thus EPN(G)=r. An edge function f: E$\to$P and its corresponding edge product function F which make $G\cup rK_2$ an edge product graph are called an optimal edge function and an optimal edge product function of G respectively.

The complete bipartite graph $K_{(p,\ q)}$ is the graph (X, Y, E) with p vertices in X and q vertices in Y with (p+q) vertices and (pq) edges. If $G=K_{(p,\ q)}\cup rK_2$ is a graph with $EPN(K_{p,\ q})=r$ where $r \geq 1$ and $2 \leq p \leq q$. Let the vertex set and the edge set of $K_{(p,q)}$ be $V_1=\{u_1, u_2, \ldots , u_p, v_1, v_2,\ldots,v_q\}$ and $E_1=\{u_iv_j: 1\leq i\leq p$ and $1\leq j\leq q\}$ respectively. The mapping f: E $\to$ P is the optimal edge function and F is its corresponding optimal edge product function of f. The optimal edge function f: E $\to$ P can be restricted to the edge set E_1 and is represented by a matrix $(a_{i,\ j})$ of order (p×q), where $a_{i,\ j} = f(u_iv_j)$ for $1\leq i\leq p$ and $1\leq j\leq q$. Then the matrix $(a_{i,\ j})$ is known as edge function matrix of the complete bipartite graph $K_{(p,\ q)}$. The row product and the column product of the optimal edge product function F is defined by

$F(u_i) = i^{th}$ row product for $1\leq i\leq p$ and

$F(v_j) = j^{th}$ column product for $1\leq j\leq q$

Since u_i and v_j are the vertices for $1\leq i\leq p$ and for $1\leq j\leq q$. The i^{th} row elements are the labelling of the edges incident on u_i and the j^{th} column elements are the labelling of the edges incident on v_j.

Let $U_{p,q}$ be a matrix of order p×q. Then the two power unit product matrix $U_{p,\ q}$ is constructed by:

$u_{i,\ j} = 2^{(i - 1)+(j - 1)(p - 1)}$ for $1\leq i\leq(p-1)$ and for $1\leq j\leq(q-1)$

$u_{(p\ ,j)} = \{1/2^{(j - 1)(p - 1)}\prod\{2^{k - 1}: 1\leq k\leq(p-1)\}\}$

for $1 \leq j \leq (q-1)$ for $1 \leq j \leq (q-1)$]

$u_{(i, q)} = \{1/2^{i-1} \prod \{2^{(k-1)(p-1)} : 1 \leq k \leq (q-1)\}\}$

for $1 \leq i \leq (p-1)$]

$u_{(p, q)} = \prod \{2^{k-1} : 1 \leq k \leq (p-1)(q-1)\}$

Lemma: 2.2

If $U_{p,q}$ is a two power unit product matrix of order $(p \times q)$. Then the product of row elements and column elements of the matrix $U_{p,q}$ are equal to one and the product of a collection of distinct elements of $U_{p,q}$ are one then they form a row or a column.

III. Main Result

Theorem: 3.1 If $K_{(3, 3q)}$ is a complete bipartite graph then prove that

$$EPN(K_{3,3q}) = \begin{cases} 2 & \text{if q is even} \\ 3 & \text{if q is odd for all } q \geq 2 \end{cases}$$

Proof: Let $K_{(3, 3q)}$ be a complete bipartite graph for $q \geq 2$. Then $3q \neq 3$ and $EPN(K_{3, 3q}) \geq 2$.

Then there may arise two cases.

Case (i) when q is even

Let $G = K_{(3, 3q)} \cup 2K_2$ be an edge product graph with the vertex set $V = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_{3q}, w_1, w_2, w_3, w_4\}$ and the edge set $E = \{u_i v_j : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3q\} \cup \{w_1 w_2, w_3 w_4\}$. If $x = 2^{6q}$ then we construct the edge product matrix $A_{(3 \times 3q)} = (a_{(i, j)})$ as:

$a_{(i, j)} = x^{(3q-1)} \times 2^{(i-1)+2(j-1)}$

for $1 \leq i \leq 2$ and $1 \leq j \leq (3q-1)$

$a_{(3, j)} = x^2 / [3(2^{2(j-1)})]$ for $1 \leq j \leq (3q-1)$

$a_{(i, 3q)} = x^{(3q-1)} / 2^{(i-1)} \prod \{2^{2(k-1)} : 1 \leq k \leq (3q-1)\}$

for $1 \leq i \leq 2$ and

$a_{(3, 3q)} = x^{(3q-1)(3q-2)} \times \prod \{2^{(k-1)} : 1 \leq k \leq 2(3q-1)\}$

The elements of P are $\{a_{(i, j)} : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3q\} \cup \{x^{6q}, x^{(3q-1)3q}\}$ and the mapping $f: E \rightarrow P$ is an edge function which is defined by $f(u_i v_j) = a_{(i, j)}$ for $1 \leq i \leq 3$ and $1 \leq j \leq 3q$; $f(w_1 w_2) = x^{6q}$ and $f(w_3 w_4) = x^{(3q-1)3q}$

The corresponding edge product function F is defined by $F(u_i) = x^{(3q-1)3q}$ for $1 \leq i \leq 3$. It is the value of $F(v_{3q}) = F(w_3) = F(w_4) = f(w_3 w_4)$ and $F(v_j) = x^{6q}$ for $1 \leq j \leq (3q-1)$ which is equal to $F(w_1) = F(w_2) = f(w_1 w_2)$. Hence F is into P. Thus the edge function f and its corresponding edge product function F, the given graph $K_{(3, 3q)} \cup 2K_2$ is an edge product graph for even integers $q \geq 2$. Therefore $EPN(K_{3, 3q}) \leq 2$. Thus we get the required result that $EPN(K_{3, 3q}) = 2$ where $q \geq 2$ is an even integer.

Case (ii) when q is odd

To prove that $EPN(K_{3, 3q}) = 3$ for all odd integers q. Suppose $EPN(K_{3, 3q}) = 2$ for some odd integer $q > 1$. Then the graph $G = K_{(3, 3q)} \cup 2K_2$ for some odd integer $q \geq 2$ is an edge product graph. Let $V = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_{3q}, w_1, w_2, w_3, w_4\}$ be the vertex set and $E = \{u_i v_j : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3q\} \cup \{w_1 w_2, w_3 w_4\}$ be the edge set of G. Let $f : E \rightarrow P$ be an optimal edge function and F be its corresponding optimal edge product function. Since F is an outer edge product function, the range set of $F = \{f(w_1 w_2), f(w_3 w_4)\}$. Suppose $f(w_1 w_2) = x$, $f(w_3 w_4) = x$ and then using the equation $\prod \{F(u_i) : 1 \leq i \leq 3\} = \prod \{F(v_j) : 1 \leq j \leq 3q\}$.

Let us take the vertex sets $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2, \ldots, v_{3q}\}$ then we have the following four cases:

Case (a) when $F(V_1) = \{x\}$ and $F(V_2) = \{y\}$

We have $F(u_i) = x$ for $1 \leq i \leq 3$ and

$F(v_j) = y$ for $1 \leq j \leq 3q$.

Therefore $x^3 = y^{3q} \Rightarrow x = y^q \Rightarrow$ y is a divisor of x which is a contradiction.

Case (b) when $F(V_1) = \{x, y\}$ and $F(V_2) = \{y\}$

Consider that $F(u_1) = F(u_2) = x$ and $F(u_3) = y$.

Then $x^2 \times y = y^{3q} \Rightarrow x^2 = 2^{3q-1} \Rightarrow$ y is a divisor of x, since q is an odd integer.

Suppose $F(u_1) = x$ and $F(u_2) = F(u_3) = y$.

Then we have $x \times y^2 = y^{3q} \Rightarrow x = y^{3q-2} \Rightarrow$ y is a divisor of x which is a contradiction

Case (c) when $F(V_1) = \{x\}$ and $F(V_2) = \{x, y\}$

$$\text{Let } F(v_j) = \begin{cases} x & \text{for } 1 \leq j \leq r \\ y & \text{for } (r+1) \leq j \leq 3q \end{cases}$$

Therefore $x^3 = x^r \times y^{3q-r}$

Hence $x^{(3-r)} = y^{(3q-r)} \Rightarrow x^{3-r} > 1$ since $y^{3q-r} > 1$

$\Rightarrow 3 - r > 1$

$\Rightarrow r = 1$ or $r = 2$

When $r = 1 \Rightarrow x^3 = x \times y^{(3q-1)} \Rightarrow$ y is a divisor of x since q is odd

When $r = 2 \Rightarrow x^3 = x^2 \times y^{(3q-2)} \Rightarrow$ y is a divisor of x which is a contradiction.

Case (d) when $F(V_1) = \{x, y\}$ and $F(V_2) = \{x, y\}$

Let $F(u_1) = x$, $F(u_2) = F(u_3) = y$

$$\text{Let } F(v_j) = \begin{cases} x & \text{for } 1 \leq j \leq r \\ y & \text{for } (r+1) \leq j \leq 3q \end{cases}.$$

Therefore $x \times y^2 = x^r \times y^{(3q-r)} \Rightarrow x^{(1-r)} = y^{(3q-r-2)}$

$\Rightarrow 1 - r > 1$ and

$3q - r - 2 > 1$

This is impossible since $r \geq 1$. Hence in all the cases we arrive at a contradiction.

Therefore, $EPN(K_{(3, 3q)}) \geq 3$ for all odd integers $q \geq 3$.

To prove that $EPN(K_{(3, 3q)}) = 3$ for odd integers $q \geq 3$ by defining an appropriate edge function. Consider the graph $G = K_{(3, 3q)} \cup 3K_2$. Let $V = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_{3q}, w_1, w_2, w_3, w_4, w_5, w_6\}$ be the vertex set and $E = \{u_i v_j : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3q\} \cup \{w_1 w_2, w_3 w_4, w_5 w_6\}$ be the edge set of G. If $x = 2^{6q}$ then the edge function matrix $A_{(3 \times 3q)} = (a_{(i, j)})_{3 \times 3q}$ is constructed as follows:

$a_{(i, j)} = x \times 2^{(i-1)+2(j-1)}$ for $1 \leq i \leq 2$ and $1 \leq j \leq (3q-1)$

$a_{(3, j)} = x / 3(2^{2(j-1)})$ for $1 \leq j \leq (3q-1)$

$a_{(i, 3q)} = x^{(9q-2)} / 2^{(i-1)} \prod \{2^{2(k-1)} : 1 \leq k \leq 2(3q-1)\}$

for $1 \leq i \leq 2$

ISSN: 2278 – 7798
International Journal of Science, Engineering and Technology Research (IJSETR)
Volume 5, Issue 4, April 2016

$a_{(3,\,3q)} = x^{(9q-2)} \times \prod\{2^{(k-1)} : 1 \le k \le 2(3q-1)\}$
If $P=\{a_{(i,\,j)}: 1\le i\le 3 \text{ and } 1\le j \le 3q\}\cup\{x^3, x^{(12q-3)}, x^{3(9q-2)}\}$
then the edge function $f: E \to P$ is defined by
$f(u_iu_j) = a_{(i,\,j)}$ for $1 \le i \le 3$ and $1 \le j \le 3q$
$f(w_1w_2) = x^3$, $f(w_3w_4) = x^{(12q-3)}$ and $f(w_5w_6) = x^{3(9q-2)}$
The corresponding edge product function F is given as:
$F(u_i) = F(w_3) = F(w_4) = f(w_3w_4) = x^{(12q-3)}$ for $1 \le i \le 3$
$F(v_j) = F(w_1) = F(w_2) = f(w_1w_2) = x^3$ for $1 \le j \le (3q-1)$
and $F(v_{3q}) = F(w_5) = F(w_6) = f(w_5w_6) = x^{3(9q-2)}$
Therefore F is into P.
Thus with the edge function f and the corresponding edge product function F, $K_{(3,3q)}\cup 3K_2$ for all odd integers $q \ge 3$ becomes an edge product graph. Hence $EPN(K_{3,3q}) \le 3$ a we get the result that $EPN(K_{3,3q}) = 3$ for all odd integers $q \ge 3$.

Example: 3.2
Let $K_{(3,3q)}$ be a complete bipartite graph. Then the edge function matrix A of order (3×6) is as follows:

$$\begin{bmatrix} 2^{60} & 2^{62} & 2^{64} & 2^{66} & 2^{68} & 2^{40} \\ 2^{61} & 2^{63} & 2^{65} & 2^{67} & 2^{69} & 2^{35} \\ 2^{23} & 2^{19} & 2^{15} & 2^{11} & 2^{7} & 2^{285} \end{bmatrix}$$

Therefore the graph $K_{(3,6)}\cup 2K_2$ is an edge product graph and $EPN(K_{3,6}) = 2$, is illustrated in figure 1.

Example: 3.3
Let $K_{(3,\,9)}$ be a complete bipartite graph. The edge function matrix A of order (3×9) of this graph is given below:

$$\begin{bmatrix} 2^{18} & 2^{20} & 2^{22} & 2^{24} & 2^{26} & 2^{28} & 2^{30} & 2^{32} & 2^{394} \\ 2^{19} & 2^{21} & 2^{23} & 2^{25} & 2^{27} & 2^{29} & 2^{31} & 2^{33} & 2^{386} \\ 2^{17} & 2^{13} & 2^{9} & 2^{5} & 2^{1} & (1/2^{3}) & (1/2^{7}) & (1/2^{11}) & 2^{570} \end{bmatrix}$$

The graph $K_{(3,9)}\cup 3K_2$ is an edge product graph and $EPN(K_{3,9}) = 3$ is shown in figure 2.

IV. CONCLUSION

It is concluded that we showed the existence of edge product number of $K_{3,3q}$ graph if q is even or odd for $q \ge 3$.

ACKNOWLEDGEMENT

The author is thankful to the reviewers for their valuable suggestions to enhance the quality of the article.

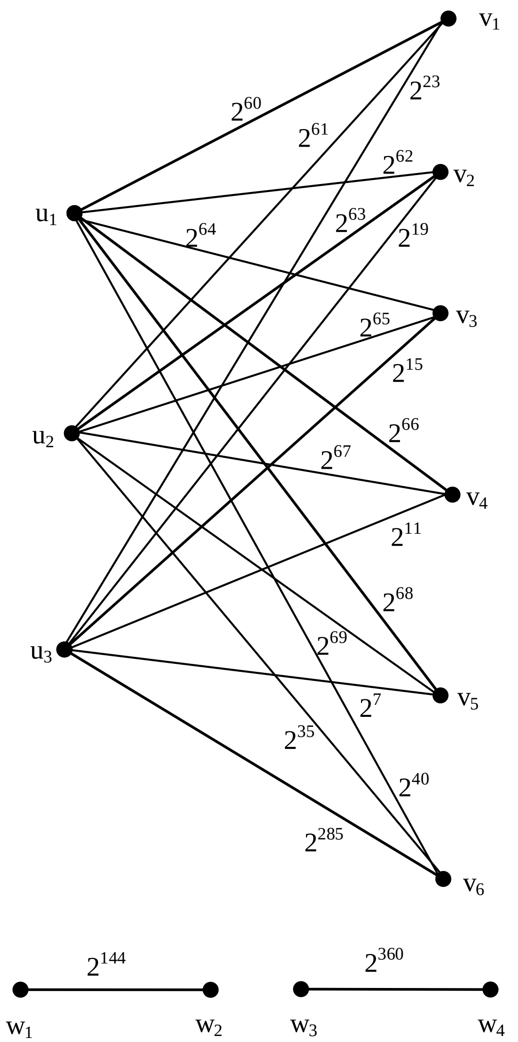

Figure 1

REFERENCES

[1] David M. Burton, Elementary Number Theory, Tata McGraw – Hill, 2006
[2] A. Kotzig, A. Rosa, Magic valuations of finite graphs, Canad. Math. Bull. 13 (1970) 451-461
[3] F. Harary, Sum graphs over all the integers, Discrete Math., 124 (1994) 99-105
[4] J. A. Gallian, A dynamic survey of graph labelling. The Electronic J. Combinatory, 5(2005) #DS6
[5] J. P. Thavamani, D.S.T. Ramesh, Edge product number of complete graphs, IJ of Mathematics & Engineering with Computers, Vol.3. No.2, (2012) 9-16

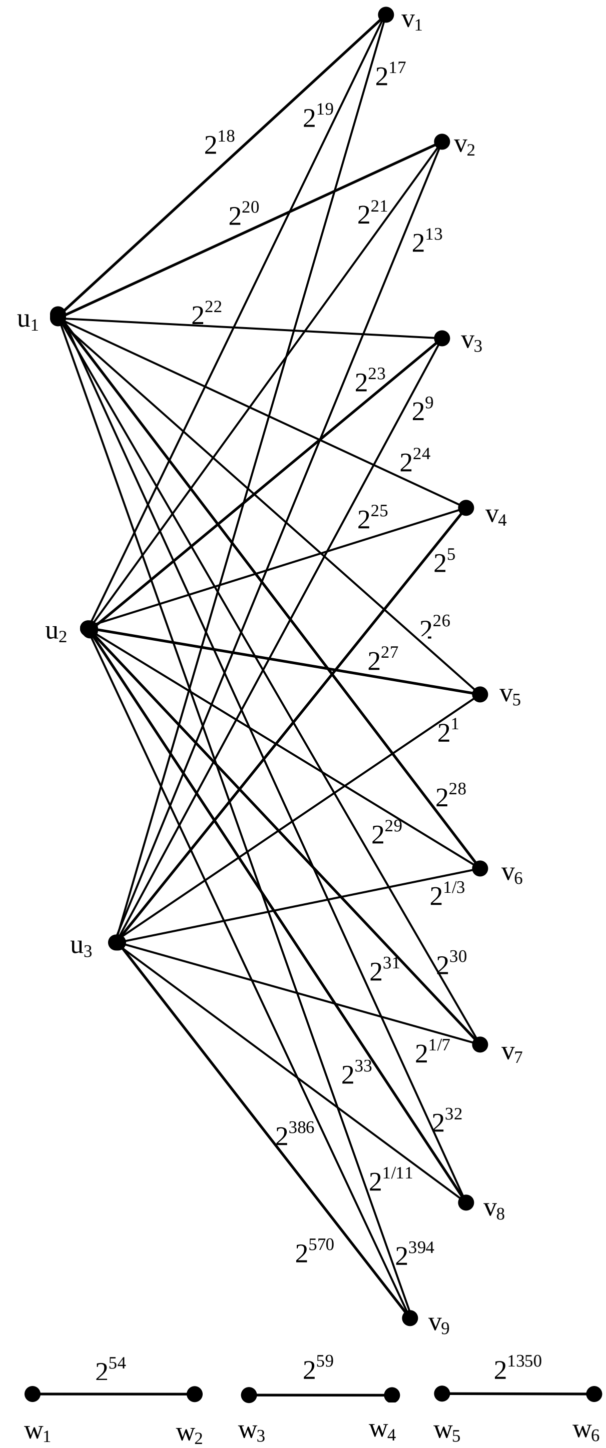

Figure 2

AUTHOR'S BRIEF BIOGRAPHY:

Prof. J. P. Thavamani:

She is a Professor in Department of Mathematics, M.E.S. College, Nedumkandam.

Her educational qualification is M.Sc.,M.Phil, PGDAOR, Ph.D. She is doing research in graph theory. She presented and published more than 15 papers in various National and International Conferences/Journals/Seminars. She is a life member of Kerala Mathematical Association. She is an Additional Chief Superintendent for the conduct of University Examinations and the internal co-coordinator of the college.