3 - sum cordial labeling of some graphs

N. Deepa\textsuperscript{1} and R. Sridevi\textsuperscript{2}

\textsuperscript{1}M.Phil Scholar, PG and Research Department of Mathematics, Sri S.R.N.M.College, Sattur-626 203, India.

\textsuperscript{2}PG and Research Department of Mathematics, Sri S.R.N.M.College, Sattur-626 203, India.

Abstract

In this paper, we proved that wheel, helm, fan, n - pan, triangular snake and crown are 3 - sum cordial graphs.

Keywords: Cordial labeling, Sum cordial labeling, 3 - Sum cordial labeling.

1 INTRODUCTION

In this paper, all graphs are finite, simple and undirected. Let $G = (V,E)$ be a graph with $|V| = p$ vertices and $|E| = q$ edges. For all other terminology and notations we follow Harary \cite{1}. A detailed study on variety of applications on graph labeling is carried out in Bloom and Golomb \cite{2}. Cordial graphs were first introduced by Cahit \cite{3} as a weaker version of both graceful graphs and harmonious graphs.

Sum cordial labeling was introduced by J. Shiama \cite{4}. In this paper, we investigate 3 - sum cordial labeling of some graphs like wheel, helm, fan, n - pan, triangular snake and crown.

2 PRELIMINARIES

Definition 2.1. A mapping $f : V (G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$. The induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0)$, $v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and $e_f(0)$, $e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^*$.

Definition 2.2. A labeling $f$ of $G$ where $N = \{0, 1\}$ and the induced edge labeling $\bar{f}$ is given by $\bar{f} (u, v) = |f(u) - f(v)|$, $\bar{N} = \{0, 1\}$. We call such a labeling cordial, if the following condition is satisfied: $|v_f(1) - v_f(0)| \leq 1$, $|e_f(1) - e_f(0)| \leq 1$, where $v_f(i)$ and $e_f(i)$, $i = \{0, 1\}$ is the number of vertices and edges of $G$ respectively, with label $i$ (under $f$ and $\bar{f}$ respectively). A graph $G$ is called cordial, if it admits a cordial labeling.

Definition 2.3. A binary vertex labeling of a graph $G$ with induced edge labeling $\bar{f} : E(G) \rightarrow \{0, 1\}$ defined by $\bar{f}(uv) = (f(u) + f(v))(mod \ 2)$ is called sum cordial labeling, if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is sum cordial, if it admits sum cordial labeling.

Definition 2.4. Let $f$ be a map from $V (G)$ to $\{0, 1, 2\}$. For each edge $uv$, assign the label
(f(u) + f(v))(mod 3). The map f is called 3-
sum cordial labeling of G, if
| v_f(i) − v_f(j) | ≤ 1 and | e_f(i) − e_f(j) | ≤ 1
for i ≠ j and i, j ∈ {0, 1, 2}. Any graph which
satisfies the 3 - sum cordial labeling is called
3 - sum cordial graph.

**Definition 2.5.** The wheel graph \( W_n \) is
defined as the join of \( K_1 + C_n \). The vertex
corresponding to \( K_1 \) is said to be apex
vertex, the vertices corresponding to cycle
are called rim vertices. The edges
corresponding to cycle are called the rim
edges and edges joining apex and vertices of
the cycle are called spoke edges.

**Definition 2.6.** The helm \( H_n \) is the graph
obtained from a wheel \( W_n \) by attaching a
pendant edge to each rim vertex.

**Definition 2.7.** The fan \( F_n \) \( (n ≥ 2) \) is
obtained by joining all vertices of the path
\( P_n \) to a further vertex called the center, and
contains \( n + 1 \) vertices and \( 2n − 1 \) edges.

**Definition 2.8.** The n - pan graph is the
graph obtained by joining a cycle graph \( C_n \)
to a singleton graph \( K_1 \) with a bridge.

**Definition 2.9.** The triangular snake \( T_n \) is
obtained from the path \( P_n \) by replacing every
edge of a path by a triangle \( C_3 \).

**Definition 2.10.** The crown \( (C_n ∩ K_1) \) is
obtained by joining a pendant edge to each
vertex of cycle \( C_n \).

3 MAIN RESULTS

**Theorem 3.1.** Any wheel \( W_n \) \( (n ≥ 3) \) is a 3 -
sum cordial graph.

**Proof.** Let \( G = W_n \) be a graph with the
vertices \( v, v_1, v_2, ..., v_n \).
Then \( | V(G) | = n + 1 \) and \( | E(G) | = 2n \).
We define \( f: V(G) → \{0, 1, 2\} \) as follows :

\[
f(v) = \begin{cases} 
0, & \text{if } i \equiv 1 \mod 3 \\
1, & \text{if } i \equiv 2 \mod 3 \\
2, & \text{if } i \equiv 0 \mod 3 
\end{cases}
\]

Thus, \( | v_f(i) − v_f(j) | ≤ 1 \) and
\( | e_f(i) − e_f(j) | ≤ 1 \) for \( i ≠ j \) and
\( i, j ∈ \{0, 1, 2\} \).
Therefore, \( W_n \) has 3 - sum cordial labeling
for \( n ≥ 3 \).
Hence, \( W_n \) \( (n ≥ 3) \) is a 3 - sum cordial graph.

**Illustration 1.** 3 - sum cordial labeling of
\( W_6 \) is shown in the figure 1.

![Figure 1](image)

**Theorem 3.2.** Any helm \( H_n \) \( (n ≥ 3) \) is a 3 -
sum cordial graph.

**Proof.** Let \( G = H_n \) be a graph. Let \( v \) be an
apex vertex and \( v_1, v_2, ..., v_n \) are rim vertices.
we denote the pendant vertices by
\( w_1, w_2, ..., w_n \).
Then \( | V(G) | = 2n + 1 \) and \( | E(G) | = 3n \).
We define \( f: V(G) → \{0, 1, 2\} \) as follows :

\[
f(v) = \begin{cases} 
0, & \text{if } n \equiv 0, 2 \mod 3 \\
1, & \text{if } 1 ≤ i, j ≤ n
\end{cases}
\]
\[
f(v_i) = f(w_j) = \begin{cases} 
0, & \text{if } i, j \equiv 1 \text{ (mod 3)} \\
1, & \text{if } i, j \equiv 2 \text{ (mod 3)} \\
2, & \text{if } i, j \equiv 0 \text{ (mod 3)} 
\end{cases}
\]

**Case (ii) :** \(n \equiv 1 \text{ (mod 3)}\)

\[
f(v) = 1, f(w_n) = 2 \text{ and for } 1 \leq i \leq n \text{ and } 1 \leq j \leq n - 1,
\]

\[
f(v_i) = f(w_j) = \begin{cases} 
0, & \text{if } i, j \equiv 1 \text{ (mod 3)} \\
1, & \text{if } i, j \equiv 2 \text{ (mod 3)} \\
2, & \text{if } i, j \equiv 0 \text{ (mod 3)} 
\end{cases}
\]

In both cases, \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\) for \(i \neq j\) and \(i, j \in \{0, 1, 2\}\).

Therefore, \(H_n\) has 3 - sum cordial labeling for \(n \geq 3\).

Hence, \(H_n\) \((n \geq 3)\) is a 3 - sum cordial graph.

**Illustration 2.** 3 - sum cordial labeling of \(H_3\) is shown in the figure 2.

**Theorem 3.3.** Any fan \(F_n\) \((n > 2)\) is a 3 - sum cordial graph.

**Proof.** Let \(G = F_n\) be a graph with the vertices \(v, v_1, v_2, ..., v_n\).

Then \(|V(G)| = n + 1\) and \(|E(G)| = 2n - 1\).

We define \(f : V(G) \rightarrow \{0, 1, 2\}\) as follows :

\[
f(v) = 1 \text{ or } 2 \text{ and for } 1 \leq i \leq n
\]

\[
f(v_i) = \begin{cases} 
0, & \text{if } i \equiv 1 \text{ (mod 3)} \\
1, & \text{if } i \equiv 2 \text{ (mod 3)} \\
2, & \text{if } i \equiv 0 \text{ (mod 3)} 
\end{cases}
\]

Thus, \(|v_f(i) - v_f(j)| \leq 1\) and \(|e_f(i) - e_f(j)| \leq 1\) for \(i \neq j\) and \(i, j \in \{0, 1, 2\}\).

Therefore, \(F_n\) has 3 - sum cordial labeling for \(n > 2\).

Hence, \(F_n\) \((n > 2)\) is a 3 - sum cordial graph.

**Note 1.** For \(n = 2\), fix \(f(v) = 2\).
Illustration 4. 3 - sum cordial labeling of $F_5$ is shown in the figure 4.

![Figure 4](image_url)

Illustration 5. 3 - sum cordial labeling of $F_2$ is shown in the figure 5.

![Figure 5](image_url)

Theorem 3.4. Any $n$ - pan ($n \geq 3$) is a 3 - sum cordial graph.

Proof. Let $G$ be a graph with the vertices $v, v_1, v_2, ..., v_n$.
Then $|V(G)| = n + 1$ and $|E(G)| = n + 1$.
We define $f : V(G) \rightarrow \{0, 1, 2\}$ as follows :

Case (i) : $n \equiv 0, 1 \ (mod \ 3)$

$$f(v) = \begin{cases} 0, \text{ if } n \equiv 0 \ (mod \ 3) \\ 2, \text{ if } n \equiv 1 \ (mod \ 3) \end{cases}$$

Case (ii) : $n \equiv 2 \ (mod \ 3)$

Subcase (i) : $n$ is odd

$$f(v) = 2 \text{ and for } 1 \leq i \leq n$$

$$f(v_i) = \begin{cases} 0, \text{ if } i \equiv 1, 2 \ (mod \ 3) \\ 1, \text{ if } i \equiv 3, 4 \ (mod \ 3) \\ 2, \text{ if } i \equiv 5, 0 \ (mod \ 3) \end{cases}$$

Subcase (ii) : $n$ is even

$$f(v) = 2, \text{ fix}$$

$$f(v_i) = \begin{cases} 0, \text{ if } 1 \leq i \leq 3 \\ 1, \text{ if } 4 \leq i \leq 6 \\ 2, \text{ if } 7 \leq i \leq 9 \end{cases}$$

and for $10 \leq i \leq n$

$$f(v_i) = \begin{cases} 0, \text{ if } i \equiv 4, 5 \ (mod \ 3) \\ 1, \text{ if } i \equiv 0, 1 \ (mod \ 3) \\ 2, \text{ if } i \equiv 2, 3 \ (mod \ 3) \end{cases}$$

In the above cases, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for $i \neq j$ and $i, j \in \{0, 1, 2\}$.
Therefore, $n$ - pan has 3 - sum cordial labeling for $n \geq 3$.
Hence, $n$ - pan ($n \geq 3$) is a 3 - sum cordial graph.

Illustration 6. 3 - sum cordial labeling of 4 - pan is shown in the figure 6.
Theorem 3.6. The triangular snake $T_n$, $(n \geq 2)$ is a 3 - sum cordial graph.

Proof. Let $v_1, v_2, ..., v_n$ and $e_1, e_2, ..., e_{n-1}$ be the vertices and edges of the path $P_n$ respectively.

To construct a triangular snake $T_n$ from the path $P_n$, join $v_i$ and $v_{i+1}$ to new vertex $v'_i$ by edges $e_{2i-1} = v_i v'_i$ and $e_{2i} = v_i v'_i$ for $i = 1, 2, ..., n-1$.

Then $|V(T_n)| = 2n - 1$ and $|E(T_n)| = 3n - 3$.

We define $f: V(T_n) \to \{0, 1, 2\}$ as follows:

For $1 \leq i \leq n$

\[ f(v_i) = \begin{cases} 
0, & \text{if } i \equiv 1 \pmod{3} \\
1, & \text{if } i \equiv 2 \pmod{3} \\
2, & \text{if } i \equiv 0 \pmod{3}
\end{cases} \]

and for $1 \leq i \leq n-1$

\[ f(v'_i) = \begin{cases} 
2, & \text{if } i \equiv 1 \pmod{3} \\
0, & \text{if } i \equiv 2 \pmod{3} \\
1, & \text{if } i \equiv 0 \pmod{3}
\end{cases} \]

Thus, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for $i \neq j$ and $i, j \in \{0, 1, 2\}$.

Therefore, $T_n$ has 3 - sum cordial labeling for $n \geq 2$.

Hence, $T_n (n \geq 2)$ is a 3 - sum cordial graph.

Illustration 8. 3 - sum cordial labeling of $T_5$ is shown in the figure 8.

Theorem 3.7. The crown $C_n \otimes K_1 (n \geq 3)$ is a 3 - sum cordial graph.

Proof. Let $G = C_n \otimes K_1$ be a graph. Let $v_1, v_2, ..., v_n$ be the vertices of the cycle $C_n$ and $w_1, w_2, ..., w_n$ be the pendant vertices of $C_n \otimes K_1$.

Then $|V(G)| = 2n$ and $|E(G)| = 2n$.

We define $f: V(G) \to \{0, 1, 2\}$ as follows:

Case (i) : $n \equiv 0 \pmod{3}$

For $1 \leq i, j \leq n$
Case (ii) : $n \equiv 1, 2 \pmod{3}$

Subcase (i) : $n$ is odd

For $1 \leq i, j \leq n$

\[
    f(v_i) = f(w_j) = \begin{cases} 
        0, & \text{if } i, j \equiv 1 \pmod{3} \\
        1, & \text{if } i, j \equiv 2 \pmod{3} \\
        2, & \text{if } i, j \equiv 0 \pmod{3}
    \end{cases}
\]

Subcase (ii) : $n$ is even

\[
    f(w_n) = 2
\]

For $1 \leq i \leq n$

\[
    f(v_i) = \begin{cases} 
        0, & \text{if } i \equiv 1 \pmod{3} \\
        1, & \text{if } i \equiv 2 \pmod{3} \\
        2, & \text{if } i \equiv 0 \pmod{3}
    \end{cases}
\]

and for $1 \leq j \leq n - 1$

\[
    f(w_j) = \begin{cases} 
        2, & \text{if } j \equiv 1 \pmod{3} \\
        0, & \text{if } j \equiv 2 \pmod{3} \\
        1, & \text{if } j \equiv 0 \pmod{3}
    \end{cases}
\]

In the above cases, $|v_f(i) - v_f(j)| \leq l$ and $|e_f(i) - e_f(j)| \leq l$ for $i \neq j$ and $i, j \in \{0, 1, 2\}$.

Therefore, the crown $C_n \circ K_1$ has 3 - sum cordial labeling for $n \geq 3$.

Hence, the crown $C_n \circ K_1$ ($n \geq 3$) is a 3 - sum cordial graph.

Illustration 9. 3 - sum cordial labeling of $C_3 \circ K_1$ is shown in the figure 9.

References
