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Symmetric Digraphs Space

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Abstract— **In this paper we defined some space of digraphs. We also obtain the dimensional of these vector spaces. These vector spaces make it possible to use techniques of linear algebra in studying the graph.**

Index Terms — **vector space; vertex space; edge space; digraph; symmetric digraph; dimensional**

I. INTRODUCTION

In the mathematical discipline of graph theory, the edge space and vertex space of a graph are vector spaces defined in terms of the edge and vertex sets, respectively. Let $G = (V; E)$ be a finite symmetric digraph.$with\ E(G) = \{e1, \dots, em\}$. The edge space of G, denoted by $\mathcal{E}_{sd}(G)$,is a vector space over the field $\mathbb{Z} = \{-1,0,1\}$ where vectors are subsets of $E(G)$, addition of vectors is defined by $P + Q = P \Delta Q = (P - Q) \cup (Q - P)$, the symmetric difference of sets P and Q, and scalar multiplication is defined by $c.P = P$ if $c = 1$ and $c.P = \emptyset$ if c = 0 and $c.P = -P$ if $c = -1$ for all $P \epsilon \mathcal{E}_{sd}(G)$, $-P \epsilon \mathcal{E}_{sd}(G)$ since the digraph is symmetric. For other basic concepts about graph theory the readers may refer to the book Introduction to Graph Theory [2] by Chartrand and Zhang or the book graph theory by Harary [7]. For basic concepts in linear algebra the readers may refer to the book Linear Algebra and Matrix Theory by Nering [8].

II. DEFINITIONS

Graph:
A labeled graph $G = (V, E)$ is a 2-tuple consisting of a set of vertices V and a set of edges E.

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Undirected graph:
An undirected graph is one in which edges have no orientation. The edge (a, b) is identical to the edge (b, a) ,i.e., they are not ordered pairs, but set $\{u, v\}$ of vertices.

Example 1.
Consider $V = \{a, b, c\}$ and $E = \{ab, ac, bc\}$ the graph $G = (V, E)$ is shown in Figure 1.

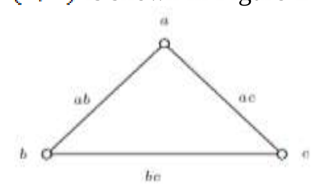

Figure 1: Graph G

*. **Digraph**:*
A digraph is an ordered pair (V, E), where is the set of vertices and E is the set of arcs or directed edge.

We usually write uv for arc (u, v). Arc uv leaves u and enters v. For any vertices u and v, when both uv and vu are arcs we call them antiparallel arcs.

The outdegree, of vertex v is the number of arcs that leave v, denoted by $od(v)$ and the indegree of vertex v is the number of arcs that enter to v, denoted by $id(v)$.

Example 2.
The graph G shown in Figure 2 is a digraph.

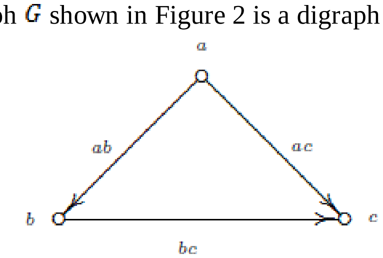

Figure 2: Digraph G

Symmetric Digraph:
A symmetric digraph is a digraph G such that if uv is an arc in G then vu is also an arc in digraph G.

Example 3:
The graph shown in Figure 3 is a symmetric digraph such that the vertex set of G is $V = \{a,b,c\}$ and the arc set of G is $E = \{ab,ba,cb,bc,ac,ca\}$.

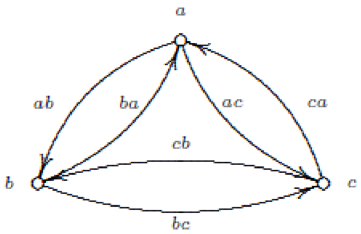

Figure 3: Symmetric Digraph G

III. Vector Space

Definition 1:
A vector space over a field F is a set V together the operations of addition $V * V \rightarrow V$ and scalar multiplication $F * V \rightarrow V$ satisfying the following properties:

- Closed under addition : For any $v,w \epsilon V$ their vector sum $v + w$ is an element of V;
- Commutativity : $u + v = v + u$ for all $u,v \epsilon V$;
- Associativity : $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u,v,w \epsilon V$ and $a,b \epsilon F$;
- Additive Identity : There exists an element $0 \epsilon V$ such that $0 + v = v$ for all $v \epsilon V$;
- Additive inverse : For every $v \epsilon V$, there exists an element $w \epsilon V$ such that $v + w = 0$;
- Closed under scalar multiplication : $av \epsilon V$ for any $v \epsilon V$ and $a \epsilon F$;
- Multiplicative Identity : $Iv = v$ for all $v \epsilon V$;
- Distributivity : $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $u;\ v \epsilon V$ and $a,b \epsilon F$.

Example 4:
Consider the set R^3 of all 3-tuples with element in R. This is vector space. Addition and scalar multiplication are defined component wise. That is for $u = (u_1,u_2,u_3), v = (v_1,v_2,v_3) \epsilon R^3$ and $a \epsilon R$ we define $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and $au = (au_1, au_2, au_3)$. It is easy to check that all properties of definition of vector space are satisfied. In particular the additive identity $0 = (0,0,0)$ and the additive inverse of u is $-u = (-u_1 -, u_2, -u_3)$

IV. Vertex Space $V_{sd}(G)$

Definition 2:
Let $G = (V,E)$ be a finite digraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. The **vertex space** of G, denoted by $V_{sd}(G)$, is a vector space over the field $\mathbb{Z}_3 = \{-1,0,1\}$ where vectors are subsets of $V(G)$, addition of vectors is defined by $R + S = R\Delta S = (R - S) \cup (S - R)$, the symmetric difference of sets R and S, and scalar multiplication is defined by $c.R = R$ if $c = 1,-1$ and $c.R = \emptyset \ if \ c = 0$.

Example 5:
Consider the digraph described in Example 1, in this example, vertex space is $V_{sd}(G) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$. If $A = \{a,c\}$ and $B = \{b,c\}$ then $A + B = A\Delta B = \{a,b\}$.

Remark 1.
The set of all subsets of $V_{sd}(G)$ together with the symmetric difference and scalar multiplication $c.R$ is the vector space over the finite field of three elements $\mathbb{Z}_3 = \{-1,0,1\}$

Proof.
To show that $V_{sd}(G)$ is vector space we check all properties of vector space in Definition 1.

- Let $R,S \epsilon V_{sd}(G)$, so $R + S = R\Delta S = (R - S) \cup (S - R)$, is a set of some of vertices of G, therefore $R + S \epsilon V_{sd}(G)$.
- Let $R,S \epsilon V_{sd}(G)$, $R + S = R\Delta S = (R - S) \cup (S - R$ but $(R - S) \cup (S - R) = (S - R) \cup (R - S$ and $= (S - R) \cup (R - S)$ = S + R so R + S = S + R for all Let $R,S \epsilon V_{sd}(G)$, then addition is commutative.
- Let $R,S,Q \epsilon V_{sd}(G)$, we will check the associativity property for $V_{sd}(G)$. In this case; ($R + S) + Q = (R\Delta S) + Q = R\Delta S\Delta Q = R\Delta(S\Delta Q) = R + (S + Q)$.
  So addition is associative.
- $\emptyset \epsilon V_{sd}(G)$. Such that $\emptyset + R = \emptyset \Delta R + R$, for all $R \epsilon V_{sd}(G)$. The empty set is additive identity for vertex space $V_{sd}(G)$.
- By definition of vector addition we have $R + R = R\Delta R = \emptyset$. Therefore for all $R \epsilon V_{sd}(G)$. the inverse is the set itself.
- Let $a \epsilon \{-1,0,1\}$ and $R \epsilon V_{sd}(G)$, if $a = -1, 1$, so by Definition 2, $a.R = R$. Hence $a.R \epsilon V_{sd}(G)$. $If \ a = 0, so \ 0.R = \emptyset \ that \ R \epsilon V_{sd}(G)$. Hence $0.R \epsilon V_{sd}(G)$. If $a = 1$ we have $1.R = R$ where $R \epsilon V_{sd}(G)$. Hence $1.R \epsilon V_{sd}(G)$.
- For all $R \epsilon V_{sd}(G)$, $1.R = R$ such that $1 \epsilon \mathbb{Z}_3$.

- Let $R, S \epsilon V_{sd}(G)$, and $a\ b \epsilon \mathbb{Z}_3 = \{-1,0,1\}$. For first part $a(R+S) = a(R \Delta S) = aR \Delta aS = aR + aS$. Now we consider all the cases for all posible value of a and b, we have $a+b = 0+0 = 0, a+b = 1+0 = 1, a+b = 0+1 = 1, a+b = 1+1 = -1, a+b = -1+0 = -1, a+b = 0(-1) = -1\ or\ a+b = -1+(-1) = 1$ for all and $a\ b \epsilon \mathbb{Z}_3$. $So\ (a+b).R = 0.R = \emptyset,\ (a+b).R = 1.R = R\ or\ (a+b).R = -1\_.R$, by Definition 2, $-1.R = R$. Therefore the vertex space $V_{sd}(G)$, of digraph G is a vector space over the finite field of 3-elements $\mathbb{Z}_3$ of all function $V \rightarrow \mathbb{Z}_3$.

V. DIMENSION OF $V_{sd}(G)$

Theorem 1.
The dimension of a vertex space on digraph G in the field $\mathbb{Z}_3$ is the number of vertices of digraph.

Proof:
Let $G = (V,E)$ is a digraph, let $V_{sd}(G)$, is the vertex space of G, if $|V| = n$, $V = \{v_1, v_2, \cdots, v_n\}$ we show that the set $H = \{\{v_1\},\{v_2\},\cdots,\{v_n\}\}$ is a basis for vector space $V_{sd}(G)$, we must show :
1. H spans $V_{sd}(G)$,
2. H is linearly independent.
We have to show that $H = \{\{v_1\},\{v_2\},\cdots,\{v_n\}\}$ spans $V_{sd}(G)$, consider the set $A = \{\emptyset,\{v_1\},\{v_2\},\cdots,\{v_n\},\{v_1\}\cup\{v_2\},\cdots,\{v_1\}\cup\{v_2\}\cup \cdots \cup\{v_n\}\} = V_{sd}(G)$
since A is all the subsets of H then H spans the vertex space $V_{sd}(G)$,
Now to show that H is linearly independent, we use vector addition and scalar multiplication, $\{v_i\} \neq \{v_j\}$, by define of scalar multiplication in $V_{sd}(G)$, when $i \neq j$ so $\{v_i\} + \{v_j\} \neq \emptyset$, and also $a\{v_i\} = \emptyset$ when $a = 0$ in $\mathbb{Z}_3$. In this case we will show $a_1\{v_1\} + \cdots + a_n\{v_n\} = \emptyset\ if\ a_i = 0$ for all $i = 1,2,\cdots,n$. Suppose otherwise that there is $a_k \neq 0$ such that $a_1\{v_1\} + \cdots + a_n\{v_n\} = \emptyset$ Then $a_k = -1,1$. Hence by Definition 2, $a_k\{v_k\} = \{v_k\} \neq \emptyset$, which is a contradiction. Then H is linearly independent, hence set H is a basis of $V_{sd}(G)$.

Illustration 1.
In the digraph $G = (V,E)$ with $V = \{a,b,c\}$ and vertex space $V_{sd}(G) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$, its basis is set $H = \{a\},\{b\},\{c\}\}$. Because sets $\{a\},\{b\}\ and\ \{c\}$ spans vertex space $V_{sd}(G)$ and they are linearly independet from the proof of theorem 1.

The set $H = \{\{v_1\},\{v_2\},\cdots,\{v_n\}\}$ is called ***standard basis*** of vector space $V_{sd}(G)$ and $V_{sd}(G)$ is an n-dimensional vector space over $\mathbb{Z}_3$.

VI. EDGE SPACE $\mathcal{E}_{sd}(G)$

Definition 3.
Let $G = (V,E)$ be a finite symmetric digraph with $E(G) = \{e_1, \cdots e_m\}$. The edge space of G, denoted by $\mathcal{E}_{sd}(G)$ is a vector space over the field $\mathbb{Z}_3 = \{-1,0,1\}$ where vectors are subsets of $E(G)$, addition of vectors is defined by $P + Q = P \Delta Q = (P - Q) \cup (Q - P)$, the symmetric difference of sets P and Q, and scalar multiplication is defined by $c.P = P$ if $c = 1$ and $c.P = \emptyset$ if $c = 0$ and $c.P = -P$ if $c = -1$ for all $P \epsilon \mathcal{E}_{sd}(G)$, $-P \epsilon \mathcal{E}_{sd}(G)$ since the digraph is symmetric.

Example 6.
Let the symmetric digraph $G = (V,E)$ with $V = \{a,b,c\}$ and $E = \{ab,ba,cb,bc,ac,ca\ \}$.its edge space is $\{\emptyset,\{ab\},\{ac\},\{bc\},\{ba\},\{ca\},\{cb\},\{ab,ac\},\{ab,ac\},\{ab,bc\},\{ac,bc\},$

Theorem 2.(Main Theorem)
The set of all subsets of $\mathcal{E}_{sd}(G)$ together with the symmetric difference and scalar multiplication $c.P$ is the vector space over the finite field of 3- elements $\mathbb{Z}_3 = \{-1,0,1\}$.

Proof.
To show that $\mathcal{E}_{sd}(G)$ is vector space we check all properties of vector space in Definition 1.

- Let $P, Q \epsilon \mathcal{E}_{sd}(G)$ so $P + Q = (P - Q) \cup (Q - P)$ is a set of some of arcs of G, therefore $P + Q \epsilon \mathcal{E}_{sd}(G)$,
- Let $P, Q \epsilon \mathcal{E}_{sd}(G)$, $P + Q = (P - Q) \cup (Q - P)$ but $(P - Q) \cup (Q - P) = (Q - P) [(P - Q)$ and $(Q - P) \cup (P - Q) = Q + P\ so\ P + Q = Q + P$ for all $P, Q \epsilon \mathcal{E}_{sd}(G)$ then addition is commutative.
- Let $P, Q, J \epsilon \mathcal{E}_{sd}(G)$ we will check the associativity property for $\mathcal{E}_{sd}(G)$. In this case; $(P + Q) + J = (P \Delta Q) + J = P \Delta Q \Delta J = P \Delta (Q \Delta J) = P + (Q + J)$.
  So addition is associative;
- $\emptyset \epsilon \mathcal{E}_{sd}(G)$ such that $\emptyset + P = \emptyset \Delta P = P$ for all $P \epsilon \mathcal{E}_{sd}(G)$. The empty set is additive identity for edge space $\mathcal{E}_{sd}(G)$;
- Let $P \epsilon \mathcal{E}_{sd}(G)$. So $P + P = (P - P) \cup (P - P) = \emptyset$. Therefore for all $P \epsilon \mathcal{E}_{sd}(G)$. The additive inverse is the set itself;
- Let $a \epsilon \{-1,0,1\}$ and $P \epsilon \mathcal{E}_{sd}(G)$. If $= -1$, so by Definition 3, $-1.P = -P$ that $-P \epsilon \mathcal{E}_{sd}(G)$Hence $-1.P \epsilon \mathcal{E}_{sd}(G)$. If $a = 0$, so $0.P = \emptyset$ that $\emptyset \epsilon \mathcal{E}_{sd}(G)$ Hence $0.P \epsilon \mathcal{E}_{sd}(G)$. If

$a = 1$ we have $1.P = P$ where $P \epsilon \mathcal{E}_{sd}(G)$. Hence $1.P \epsilon \mathcal{E}_{sd}(G)$. (closed under scalar multiplication);

- For all $\epsilon \mathcal{E}_{sd}(G)$, $1.P = P$ such that $1 \epsilon \mathbb{Z}_3$. (multiplicative Identity);
- Let $P, Q \epsilon \mathcal{E}_{sd}(G)$ and $a, b \epsilon \mathbb{Z}_3$. For first part $a(P + Q) = a(P \Delta Q) = aP\Delta aQ = aP + aQ$. Now for second part, we have $a + b = 0 + 0 = 0, a + b = 1 + 0 = 1, a + b = 0 + 1 = 1, a + b = 1 + 1 = -1, a + b = -1 + 0 = -1, a + b = 0 + (-1) = -1$ *or* $a + b = -1 + (-1) = 1$ for all $a, b \epsilon \mathbb{Z}_3$. So $(a+b).P = 0.P = \emptyset, (a+b).P = 1.P = P$ *or* $(a+b).P = -1_P = -P$.

Therefore the edge space $\mathcal{E}_{sd}(G)$ of symmetric digraph G is a vector space over the finite field of 3-elements $\mathbb{Z}_3$ of all function $E \to \mathbb{Z}_3$.

Illustration 2.
See the example 6, let $P = \{ac, ab\ bc\}$ and $Q = \{ac, ba, cb, ca\}$ so the $P + Q = (P - Q) \cup (Q - P) = (\{ac, ab, bc\} - \{ac, ba, cb, ca\}) \cup (\{ac, ba, cb, ca\} - \{ac, ab, bc\}) = \{ab, bc, ba, cb, ca\} \epsilon \mathcal{E}_{sd}(G)$
.

5.1 Dimension of $\mathcal{E}_{sd}(G)$

Theorem 2.
The dimension of a edge space of G is the number of edges of symmetric digraph.

Proof.
Let $G = (V, E)$ is a symmetric digraph, let $\mathcal{E}_{sd}(G)$ is the edge space of G, if $|E| = m$, $E(G) = \{e_1, \dots, e_m\}$. we show that the set $K = \{\{e_1\}, \dots, \{e_m\}\}$. that is subset of $\mathcal{E}_{sd}(G)$ is a basis for edge space $\mathcal{E}_{sd}(G)$. We must to show:
1. K spans $\mathcal{E}_{sd}(G)$;
2. K is linearly independent.
We have to show that $K = \{\{e_1\}, \dots, \{e_m\}\}$ spans $\mathcal{E}_{sd}(G)$. Consider the set $AB = \{\emptyset, \{e_1\}, \{e_2\}, \cdots, \{e_m\}, \{e_1\} \cup \{e_2\}, \cdots, \{e_1\} \cup \{e_2\} \cup \cdots \cup \{e_m\}\} = \mathcal{E}_{sd}(G)$
Since B is all the subsets of K, then the set K spans the edge space $\mathcal{E}_{sd}(G)$.
Now to show that K is linearly independent, we use vector addition and scalar multiplication, $\{e_i\} \neq \{e_j\}$ when $i \neq j$ so $\{e_i\} + \{e_j\} = \emptyset$ and also $a\{e_i\} = \emptyset$ when $a = 0$ in $\mathbb{Z}_3$. In this case we will show $a_1\{e_1\} + \cdots + a_m\{e_m\} = \emptyset$ *if* $a_i = 0$ for all $i = 1, 2, \cdots, m$. Suppose otherwise that there is $a_k \neq 0$ such that $a_1\{e_1\} + \cdots + a_m\{e_m\} = \emptyset$ Then either if there is one k such that $a_k = -1, 1$. Hence by Definition 3 $a_k\{e_k\} = \{e_k\} \neq \emptyset$, which is a contradiction. Then H is linearly independent, hence set H is a basis of $\mathcal{E}_{sd}(G)$.

Illustration 3.
In the symmetric digraph $G = (V, E)$ with $V = \{a, b\ c\}$ and $E = \{ab, ba, cb, bc, ac, ca\ \}$.its edge space is $\{\emptyset, \{ab\}, \{ac\}, \{bc\}, \{ba\}, \{ca\}, \{cb\}, \{ab, ac\}, \{ab, ac\}, \{ab, bc\}, \{ac, bc\},$ its basis is set L = $\{\{ab\}, \{ba\}, \{cb\}, \{bc\}, \{ac\}, \{ca\}\}$. The sets $\{ab\}, \{ba\}, \{cb\}, \{bc\}, \{ac\}$ *and* $\{ca\}$ spans edge space $\mathcal{E}_{sd}(G)$ and they are linearly independent from the proof of theorem 2.

The set $K = \{\{e_1\}, \dots, \{e_m\}\}$ where $E = \{e_1, \dots, e_m\}$. , $|E| = m$ is called ***standard basis*** of vector space $\mathcal{E}_{sd}(G)$; and $\mathcal{E}_{sd}(G)$;is an m-dimensional vector space over $\mathbb{Z}_3$.
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