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A FIXED POINT THEOREM FOR QUASI-CONTRACTIONS OF D*-METRIC SPACES

By
Dr C Vijender
Dept of Mathematics,
Sreenidhi Institute of Sciece and Technology, Hyderabad.

ABSTRACT: The main aim of this paper is to prove the existence of fixed point on λ-generalized contraction of self-mapping functions on D^*-metric space.

Key Words: D*-metric space, K-contraction**,** λ-generalized contraction.

I. INTRODUCTION:

The notion of Quasi-contraction defined for selfmaps of metric spaces given by Lj. B. Ciric [3] has been extended to the selfmaps of D^*-metric spaces as follows:

(1.1)Definition: A selfmap f of a D^*-metric space $(X, D^*)$is called a ***Quasi-contraction***, if there is a number q with $0 \le q < 1$ such that

$$D^*(fx, fy, fy) \le q.\max\{\ D^*(x, y, y), D^*(x, fx, fx), D^*(y, fy, fy), D^*(x, fy, fy), D^*(y, fx, fx)\ \} \quad \textbf{(1.2)}$$

for all $x, y \in X$.

As already noted for every λ-generalized contraction is a quasi-contraction. However the following example gives a quasi-contraction f on a D^*-metric space (X, D^*) which is not a λ-generalized contraction.

II. PRELIMINARY NOTES:

Definition: Let f be a selfmap of a D^*-metric space (X, D^*) and $x \in X$, $n \ge 1$ be an integer. The ***orbit of*** x ***under*** f ***of length*** n, denoted by $O_f(x:n)$, is defined by

$$O_f(x:n) = \{x, fx, f^2x, \ldots, f^n x\}$$

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We define the diameter $\delta(A)$ of a set A in a D^*-metric space (X, D^*) by $\delta(A) = \sup_{x, y \in A} \{D^*(x, y, y)\}$

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on D^*-metric spaces:

(2.1) Lemma: Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and n be a positive integer. Then for each $x \in X$ and all integers $i, j \in \{1, 2, 3, \ldots, n\}$,

$$D^*(f^i x, f^j x, f^j x) \leq q.\delta[O_f(x:n)] < \delta[O_f(x:n)].$$

Proof: Let $x \in X$ be arbitrary, $n \geq 1$ be an integer and $i, j \in \{1, 2, 3, \ldots, n\}$. Then $f^{i-1}x, f^{j-1}x, f^i x, f^j x \in O_f(x:n)$ and since f is a quasi-contraction,

$$D^*(f^i x, f^j x, f^j x) = D^*(ff^{i-1}x, ff^{j-1}x, ff^{j-1}x)$$

$$\begin{aligned} &\leq q.\max\{D^*(f^{i-1}x, f^{j-1}x, f^{j-1}x), D^*(f^{i-1}x, f^i x, f^i x), \\ &\qquad D^*(f^{j-1}x, f^j x, f^j x), D^*(f^{i-1}x, f^j x, f^j x), \\ &\qquad D^*(f^{j-1}x, f^i x, f^i x)\} \end{aligned}$$

$$\leq q.Sup\{D^*(u, v, v,) : u, v \in O_f(x:n)\}$$

$$= q.\delta[O_f(x:n)]$$

$$< \delta[O_f(x:n)]$$

(2.2) Lemma: Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and $x \in X$, then for every positive integer n, there exists positive integer $k \le n$, such that

$$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:n\right)\right]$$

Proof: If possible assume that the result is not true. This implies that there is positive integer m such that for all $k \le m$, we have $D^*\left(x, f^k x, f^k x\right) \ne \delta\left[O_f\left(x:m\right)\right]$. Since $O_f\left(x:m\right)$ contains x and $f^k x$ for $k \le m$, it follows that

$$D^*\left(x, f^k x, f^k x\right) < \delta\left[O_f\left(x:m\right)\right]$$

Since $O_f\left(x:m\right)$ is closed, there exists $i, j \in \{1, 2, 3, \dots, m\}$ such that $D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:m\right)\right]$, contradicting the Lemma 3.2.1. Therefore

$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f\left(x:n\right)\right]$ for some $k \le n$.

In this section we prove

III. MAIN RESULT:

3.1 Theorem: Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and X is f-orbitally complete. Then f has a unique fixed point $u \in X$. In fact,

(3.2) $$u = \lim_{n \to \infty} f^n x \text{ for any } x \in X$$

and

(3.3) $$D^*\left(f^n x, u, u\right) \le \frac{q^n}{1-q} D^*\left(x, fx, fx\right) \text{ for all } x \in X,\ n \ge 1.$$

Proof: Let x be an arbitrary point of X. We claim that $\{f^n x\}$ is a Cauchy sequence in X.

Let m, n be any positive integers with $n < m$. Since f is quasi-contraction,

$$D^*\left(f^n x, f^m x, f^m x\right) = D^*\left(ff^{n-1}x, ff^{m-1}x, ff^{m-1}x\right)$$

$$\leq q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

That is,

(3.4) $$D^*\left(f^n x, f^m x, f^m x\right) \leq q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

According to the Lemma 2.2, there exists an integer k_1, with $0 \leq k_1 \leq m-n+1$, such that

(3.5) $$\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right] = D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right)$$

Using Lemma 2.1, we get

$$D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right) = D^*\left(ff^{n-2}x, f^{k_1+1} f^{n-2}x, f^{k_1+1} f^{n-2}x\right)$$

$$\leq q.\delta\left[O_f\left(f^{n-2}x : k_1+1\right)\right]$$

$$\leq q.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

(Since $k_1 + 1 \leq m-n+2$)

Thus

(3.6) $$D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right) \leq q.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

From (3.4), (3.5) and (3.6) we get

$$D^*\left(f^n x, f^m x, f^m x\right) \le q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

$$\le q^2.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

Proceeding in this manner, we obtain

$$D^*\left(f^n x, f^m x, f^m x\right) \le q^n.\delta\left[O_f\left(x : m\right)\right]$$

Using Lemma 2.2, we get

(3.7) $$D^*\left(f^n x, f^m x, f^m x\right) \le \frac{q^n}{1-q}.\delta\left[O_f\left(x : m\right)\right]$$

Letting $n \to \infty$ and since $\lim_{n \to \infty} q^n = 0$, we get that $\left\{f^n x\right\}$ is Cauchy sequence. Again X being f-orbitally complete and $\left\{f^n x\right\}$ is a Cauchy sequence in $O_f\left(x : \infty\right)$, there is a point $u \in X$ such that $u = \lim_{n \to \infty} f^n x$.

We shall now show that u is a fixed point of f.

Consider

$$D^*(u, fu, fu) \le D^*\left(u, f^{n+1}u, f^{n+1}u\right) + D^*\left(f^{n+1}u, fu, fu\right)$$

$$= D^*\left(u, f^{n+1}u, f^{n+1}u\right) + D^*\left(ff^{n}u, fu, fu\right)$$

$$\le D^*\left(u, f^{n+1}u, f^{n+1}u\right) + q.\max\{ \; D^*\left(f^{n}u, u, u\right),$$

$$D^*\left(f^{n}u, f^{n+1}u, f^{n+1}u\right), D^*(u, fu, fu),$$

$$D^*\left(f^{n}u, fu, fu\right), D^*\left(u, f^{n+1}u, f^{n+1}u\right) \; \}$$

$$\le D^*\left(u, f^{n+1}u, f^{n+1}u\right) + q.\left\{D^*\left(f^{n}u, f^{n+1}u, f^{n+1}u\right) + D^*\left(f^{n}u, u, u\right)\right.$$

$$\left. + D^*(u, fu, fu) + D^*\left(f^{n+1}u, u, u\right)\right\}$$

Letting $n \to \infty$ and since $\lim_{n \to \infty} f^{n}x = u$, we get

$D^*(u, fu, fu) = 0$ and hence $fu = u$, showing that u is fixed point of f.

To prove the uniqueness, let u, u' be two fixed points of f. That is, $fu = u,\; fu' = u'$

$$D^*(u, u', u') = D^*(fu, fu', fu')$$

$$\le q.\max\{ \; D^*(u, u', u'), D^*(u, fu, fu), D^*(u', fu', fu'),$$

$$D^*(u, fu', fu'), D^*(u', fu, fu) \; \}$$

$$D^*(u, u', u') \le q.\max\{ \; D^*(u, u', u'), D^*(u, fu, fu), D^*(u', fu', fu'),$$

$$D^*(u, fu', fu'), D^*(u', fu, fu) \; \}$$

That is, $D^*(u,u',u') \le q.D^*(u,u',u')$

Since $q < 1$, $D^*(u,u',u') = 0$, which implies that $u = u'$.

Letting $n \to \infty$ in (3.7) we get (3.3). This completes the proof of the theorem.

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