On αgspI-continuous function in topological space with respect to an ideal

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Abstract

In this paper, we define a new class of αgspI-continuous function in topological space with respect to an ideal and discuss their some basic properties of αgspI-continuous, αgspI-totally-continuous, totally αgspI continuous.

Keywords: αgspI-continuous, αgspI-totally-continuous, totally αgspI-continuous.

1.Introduction

Ideals in topological space have been considered since 1930. This topic has won its importance by the paper of Vaidyaathaswamy[11]. In 1990, Jankovic and Hamlett [5] introduced the notation of I-open sets in topological spaces with respect to an ideal. Abd El-Monsef et al. [1] further investigated I-open sets and I-continuous functions. In 1999, Abd El-Monsef et al [2] introduced and investigated almost I-open sets and almost I-continuous functions. In 2002, Hatir and Noiri [4] have introduced the notion of αI continuous functions and used to obtain a decomposition of continuity. In this paper we define a new class of αgspI-continuous, αgspI- totally continuous, totally αgspI-continuous in topological space with respect to an ideal and their some basic properties.

2. Preliminaries

In this section, we summarize the definitions and results which are needed in sequel. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, cl(A) and int(A) denote the closure and interior of A in (X, τ) respectively. Given a topological space (X, τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\wp(X) \rightarrow \wp(X)$, called a local function of A with respect to I and τ is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{x \in X / A \cap U \notin I$, for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau / x \in U\}$ [5]. Note that $cl^*(A) = A \cup A$ defines a Kuratowski operator for a topology $\tau^*(I)$ (also denoted by if there is no ambiguity), finer than τ. A basis $\beta(I, \tau)$ for $\tau^*(I)$ can be described as follows: $\beta(I, \tau) = \{U \setminus I : U \in \tau$ and $I \in I\}$. β is not always a topology [5]. $cl^*(A)$ and $int^*(A)$ denote the closure and interior of A in (X, τ^*) respectively.

Definition 2.1. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties.
(a) $A \in I$ and $B \subseteq A$ imply $B \in I$
(b) $A \in I$ and $B \in I$ imply $A \cup B \in I$
A topological space with respect to an ideal is denoted by (X, τ, I)

Definition 2.2. A subset A of a topological space with respect to an ideal (X, τ, I) is called
(i) I-open [6] if $A \subseteq int(A^*)$
(ii) regularI-open [9] if $A = int(cl^*(A))$

(iii) preI-open [3] if A $\subseteq$ int(cl^* (A))
(iv) αI-open [4] if A $\subseteq$ int(cl^*(int(A)))
(v) semi preI-open [4] if A $\subseteq$ cl(int(cl^*(A)))

The complement of the above mentioned open sets are their respective closed sets. The semi preI-closure (resp. preI-closure, αI-closure, regularI-closure) of a subset A of (X, τ, I) is the intersection of all semi preI-closed(resp. preI-closed, αI-closed, regularI-closed) sets containing A and is denoted by spIcl(A)(resp. pIcl(A), αIcl(A), rIcl(A)).

Definition 2.3. A subset A of a topological space with respect to an ideal (X, τ, I) is called
1. α generalized I-closed (αgI-closed)[7] if αIcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is αI-open.
2. α generalised semi pre I-closed (αgspI-closed) [7] if spIcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is α generalized I-open.

Definition 2.4. A function f : (X, τ, I) → (Y, σ, J) is called [3, 4] I-continuous (res preI-continuous, semiI-continuous, αI-continuous, semi preI-continuous, rI-continuous) if f^{-1}(V) is I-closed (res preI-closed, semiI-closed, αI-closed, semi preI-closed) in (X, τ, I) for each closed set V in (Y, σ, J).

Definition 2.5. A function f : (X, τ, I) → (Y, σ, J) is called
1. perfectly-continuous [8] if f^{-1}(V) is clopen in (X, τ, I) for each open set V in (Y, σ, J).
2. totally-continuous [8] if f^{-1}(V) is clopen in (X, τ, I) for each open set V in (Y, σ, J).

Lemma 2.6. [7]
1. Every I-closed set is αgspI-closed.
2. Every semi preI-closed set is αgspI-closed.
3. Every preI-closed set is αgspI-closed.
4. Every αI-closed set is αgspI-closed.
5. Every rI-closed set is αgspI-closed.
6. Every closed set is αgspI-closed.
7. Every semiI-closed set is αgspI-closed.

3. αgspI-continuous function

In this section we define a new class of αgspI-continuous in topological space with respect to an ideal and discuss their some basic properties.

Definition 3.1. A function f : (X, τ, I) → (Y, σ, J) is called **αgspI-continuous** if f^{-1}(V) is αgspI-closed in (X, τ, I) for each closed set V in (Y, σ, J).

Example 3.2. Let X = {a, b, c, d} = Y, τ = {X, φ, {a, c}, {d}, {a, c, d}}, σ = {Y, φ, {a}, {b}, {a, b}}. Let I = {φ, {c}, {d}, {c, d}} and J = {φ}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = a, f(b) = d, f(c) = b, f(d) = c.
αgspIcl(X, τ, I) = {X, φ, {b}, {c}, {d}, {b, c}, {a, c}, {b, d}, {c, d}, {a, b, c}, {b, c, d}, {a, b, d}}.
Then the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous.

Theorem 3.3. Every continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is continuous. Let A be a closed set in Y. Since f is continuous, f^{-1}(A) is closed in X. By lemma [2.6], f^{-1}(A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.4. The converse of the above theorem is need not be true.

Example 3.5. In example [3.2], here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not continuous because f^{-1}({b}) = {d} is not closed in X.

Theorem 3.6. Every I-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is I-continuous. Let A be a closed set in Y. Since f is I-continuous, f^{-1}(A) is I-closed in X. By lemma [2.6], f^{-1}(A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.7. The converse of the above theorem is need not be true.

Example 3.8. In example [3.2], Icl(X, τ, I) = {X, φ}. Here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not I-continuous because f^{-1}({c, d}) = {b, d} is not I-closed in X.

Theorem 3.9. Every preI-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is preI-continuous. Let A be a closed set in Y. Since f is preI-continuous, f^{-1}(A) is preI-closed in X. By lemma [2.6], f^{-1}(A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.10. The converse of the above theorem is need not be true.

Example 3.11. In example [3.2], pIcl(X, τ, I) = {X, φ}, {b, c, d}, {a, b, c}, {b, d}, {b, c}, {c}, {b}}. Here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not preI-continuous because f^{-1}({a, c, d}) = {a, b, d} is not preI-closed in X.

Theorem 3.12. Every semiI-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is semiI continuous. Let A be a closed set in Y. Since f is semiI-continuous, f^{-1}(A) is semiI-closed in X. By lemma [2.6], f^{-1}(A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.13. The converse of the above theorem is need not be true.

Example 3.14. In example [3.2], sIcl(X, τ, I) = {X, φ}, {a, b, c}, {b, d}, {d}, {b}}. Here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not semiI-continuous because
f^{-1}({b, c, d}) = {b, c, d} is not semiI-closed in X.

Theorem 3.15. Every αI-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is αI continuous. Let A be a closed set in Y. Since f is αI-continuous,
f^{-1} (A) is αI-closed in X. By lemma [2.6], f^{-1} (A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.16. The converse of the above theorem is need not be true.

Example 3.17. In example [3.2], αIcl(X, τ, I) = {X, φ}, {a, b, c}, {b, d}, {d}, {b}}. Here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not αI-continuous because
f^{-1}({a, c, d}) = {a, b, d} is not αI-closed in X.

Theorem 3.18. Every rI-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is rI continuous. Let A be a closed set in Y. Since f is rI-continuous, f^{-1}(A) is rI-closed in X. By lemma [2.6], f^{-1} (A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.19. The converse of the above theorem is need not be true.

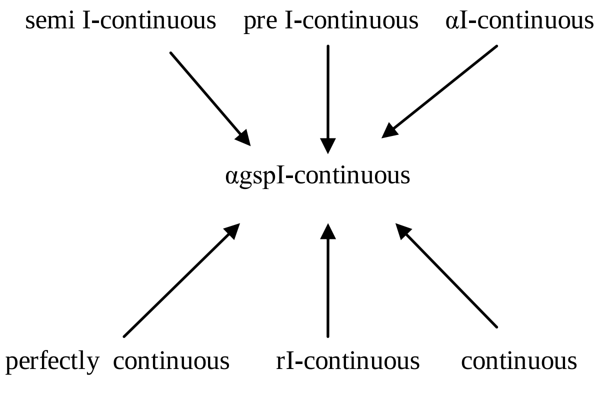
Example 3.20. Let X = {a, b, c, d} = Y, τ = {X, φ, {a, c}, {d}, {a, c, d}}, σ = {Y, φ, {b, c}}, Let I = {φ, {c}, {d}, {c, d}} and J = {φ}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = a, f(b) = c, f(c) = d, f(d) = b.
rIcl(X, τ, I) = {X, φ, {a, b, c}, {b, d}, {c}, {b, c}, {b}}. Then the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not rI-continuous because f^{-1} ({b, c, d}) = {b, c, d} is not rI-closed in X.

Theorem 3.21. Every perfectly-continuous function is αgspI-continuous.
Proof. Assume that f : (X, τ, I) → (Y, σ, J) is perfectly-continuous. Let A be a closed set in Y. Since f is perfectly-continuous, f^{-1} (A) is closed in X. By lemma [2.6], f^{-1} (A) is αgspI-closed in X. Thus f is αgspI-continuous.

Remark 3.22. The converse of the above theorem is need not be true.

Example 3.23. In example [3.20], here the inverse image of every closed set in Y is αgspI-closed in X. Therefore f is αgspI-continuous. But f is not perfectly continuous because f^{-1}({c, d}) = {b, d} is closed but not open in X.

Remark 3.24. As a consequence of the theorems [3.3, 3.6, 3.9, 3.12, 3.15,3.18, 3.21] and Examples [3.5, 3.8, 3.11, 3.14, 3.17, 3.20, 3.23] the following implicaltion diagram holds.
In this diagram, A → B means A implies B but B does not imply A.



4. αgspI-totally continuous functions

In this section we define a new class of αgspI-totally continuous in topological space with respect to an ideal and discuss their some basic properties.

Definition 4.1. A function f : (X, τ, I) → (Y, σ, J) is **αgspI-totally continuous function** if the inverse image of every αgspI-open subset of Y is clopen in X.

Theorem 4.2. A function f : (X, τ, I) → (Y, σ, J) is αgspI-totally continuous if and only if the inverse image of every αgspI-closed subset of Y is clopen in X.
Proof. Let F be any αgspI-closed set in Y. Then F^c is αgspI-open in Y. By above definition, $f^{-1}(F^c)$ is clopen in X. But $f^{-1}(F^c) = (f^{-1}(F))^c$ which is clopen in X. This implies $f^{-1}(F)$ is clopen in X.
Conversely suppose V is αgspI-open Y, then V^c is αgspI-closed in Y. By hypothesis $f^{-1}(V^c)$ is clopen in X. But $f^{-1}(V^c) = (f^{-1}(V))^c$ which is clopen in X, which implies $f^{-1}(V)$ is clopen in X. Hence the inverse image of every αgspI-open set in Y is clopen in X. Thus f is αgspI-totally continuous.

Theorem 4.3. Every αgspI-totally continuous function is totally continuous.
Proof. Suppose f : (X, τ, I) → (Y, σ, J) is αgspI-totally continuous. Let U be any open subset of Y. Since every open set is αgspI-open, U is αgspI-open in Y and also f is αgspI-totally continuous, we get $f^{-1}(U)$ is clopen in X.

Remark 4.4. The converse of the above theorem is need not be true.

Example 4.5. Let X = {a,b,c} = Y with τ = {X, φ, {a},{b,c}}. Let I = {φ, {a}}, σ = {φ, Y, {a}}, J = {φ, {a}}; αgspIo(Y,σ,J) = {Y, φ, {a}, {a, b}, {a,c}, {b}, {c}}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = a, f(b) = b, f(c) = c. Then the inverse of every open set in Y is clopen in X. Thus f is totally continuous. But f is not αgspI-totally continuous since the αgspI-open set {a,c} of Y, f^{-1}({a,c}) = {a,c} is not clopen in X.

Theorem 4.6. Every αgspI-totally continuous function is perfectly continuous
Proof. Suppose f : (X, τ, I) → (Y, σ, J) is αgspI-totally continuous. Let U be any open subset of Y. Since every open set is αgspI-open, U is αgspI-open in Y and also f is αgspI-totally continuous, we get $f^{-1}(U)$ is clopen in X. Thus f is perfectly continuous.

Remark 4.7. The converse of the above theorem is need not be true.

Example 4.8. In example [4.5], here the inverse of every open set in Y is clopen in X. Thus f is perfectly continuous. But f is not αgspI-totally continuous since the αgspI-open set {a,b} of Y, f^{-1}({a,b}) = {a,b} is not clopen in X.

Theorem 4.9. Let f : X → Y be a function, where X and Y are topological space with respect to an ideal. Then the following are equivalent.
(i) f is αgspI-totally continuous.
(ii) for each $x \in X$ and each αgspI-open set V in Y with $f(x) \in V$, there is a clopen set U in X such that $x \in U$ and $f(U) \subseteq V$.
Proof. (i) =) (ii) Suppose f is αgspI-totally continuous and V be any αgspI open set in Y containing f(x) so that $x \in f^{-1}(V)$. Since f is αgspI-totally continuous, $f^{-1}(V)$ is clopen in X. Let $U = f^{-1}(V)$, then U is clopen in X and $x \in U$. Hence $f(U) = f(f^{-1}(V)) \subseteq V$.
(ii) =) (i) Let V be αgspI-open set in Y. Let $x \in f^{-1}(V)$ be any arbitrary point. This implies $f(x) \in V$. Then there is a clopen set $f(G) \subseteq X$ containing x such that $f(G) \subseteq V$, which implies $G \subseteq f^{-1}(V)$ and $x \in G \subseteq f^{-1}(V)$.
This implies $f^{-1}(V)$ is clopen neighborhood of each of its points. Hence it is clopen set in X.
Thus f is αgspI-totally continuous.

Theorem 4.10. If a function f : (X, τ, I) → (Y, σ, J) is αgspI-totally continuous then it is continuous.
Proof. Let V be an open set in Y. Then V is αgspI-open in Y. Since f is αgspI-totally continuous, $f^{-1}(V)$ is both open and closed in X. Thus f is continuous.

Remark 4.11. The converse of the above theorem is need not be true.

Example 4.12. In example [4.5], here the inverse of every closed set in Y is αgspI-closed set in X. Thus f is continuous. But f is not αgspI-totally continuous since the αgspI-open subset {a,c} of Y, f^{-1}({a,c}) = {a,c} is not clopen in X.

Theorem 4.13. The composition of two αgspI-totally continuous function is αgspI-totally continuous.

Proof. Let f : (X, τ, I) → (Y, σ, J) and g : (Y, σ, J) → (Z, η,K) be any two αgspI-totally continuous functions. Let V be αgspI-open set in Z. Since g is αgspI-totally continuous, g^{-1} (V) is αgspI–open in Y. Also f is αgspI-totally continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}$ is clopen in X. Hence g o f is αgspI-totally continuous function.

5. Totally αgspI-continuous function

In this section we define a new class of totally αgspI-continuous in topological space with respect to an ideal and discuss their some basic properties.

Definition 5.1. A function f : (X, τ, I) → (Y, σ, J) is **totally αgspI-continuous function** if $f^{-1}(V)$ is αgspI-clopen in X, for each open set V in Y.

Example 5.2. Let X = {a,b,c} = Y with τ = {X, φ, {a},{b,c}}. Let I = {φ, {a}}, σ = {φ, Y, {a}}, J = {φ, {a}}; αgspIo(X,τ, I) = {X, φ, {a}, {a,b}, {a,c}, {b}, {c},{b,c}}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = a, f(b) = b, f(c) = c. Then the inverse of every open set in Y is αgspI-clopen in X. Thus f is totally αgspI-continuous.

Theorem 5.3. Every totally αgspI-continuous function is αgspI-continuous.
Proof. Let f : (X, τ, I) → (Y, σ, J) is totally αgspI- continuous. Let A be an open set in Y. Since f is totally αgspI-continuous, $f^{-1}(A)$ is αgspI-clopen in X. Thus f is αgspI-continuous.

Remark 5.4. The converse of the above theorem is need not be true.

Example 5.5. Let X = {a,b,c} = Y, τ = {X, φ, {a},{a,b}}. Let I = {φ, {b}}, σ = {φ, Y, {a,b},{b}},
αgspIcl(X,τ, I) = {X,φ, {b},{c},{b,c}}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = b, f(b) = a, f(c) = c.
Then the inverse image of every closed set in Y is αgspI-closed in X. Thus f is αgspI continuous. But f is not totally-αgspI continuous since the subset {b} in Y, $f^{-1}(\{b\})$ = {a} is αgspI-open but not αgspI-closed in X.

Theorem 5.6. Every αgspI-totally continuous function is totally αgspI-continuous.
Proof. Suppose f : (X, τ, I) → (Y, σ, J) is αgspI-totally continuous. Let U be any open subset of Y. Then U is αgspI-open in Y and $f^{-1}(U)$ is clopen in X. Thus $f^{-1}(U)$ is αgspI-clopen in X. Hence f is totally αgspI-continuous.

Remark 5.7. The converse of the above theorem is need not be true.

Example 5.8. In example [5.2], here the inverse of every open set in Y is αgspI-clopen in X. Thus f is totally αgspI-continuous. But f is not αgspI totally continuous because the subset {a,c} is αgspI-open in Y, $f^{-1}(\{a,c\})$ = {a,c} is not clopen in X.

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